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VOLUME 4 1961

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THE FIRST ALL-UNION CONFERENCE ON HEAT AND MASS TRANSFER

THE All-Union Conference on Heat and Mass Transfer problems was held in Minsk, B.S.S.R., U.S.S.R., June 5–9, 1961.

The Sponsors of this conference were:

The Academy of Science of the Byelorussian Soviet Socialist Republic,
The Academy of Civil Engineering and Architecture of the U.S.S.R.,
The Ministry of High and Secondary Special Education of the U.S.S.R.,
The Institute of Mechanics of the Academy of Science of the U.S.S.R.,
The Krzhyzhanovsky Power Institute.

The total number of participants was 835. Seven hundred and forty delegates came to the conference on heat and mass transfer from different parts of our country. Among the participants of the conference there were fifteen outstanding scientists who had come from eight foreign countries. One hundred and eight participants of this conference represented different industrial enterprises, 477 were sent by various research and design establishments and 155 by Institutes of Higher Education. Among the delegates there were 79 academicians, professors and doctors; 296 candidates of technical sciences, readers, senior scientific workers; 365 managers of various enterprises, shop managers and engineers. There were nine sections working where more than 300 contributors took an active part.

The President of the Academy of Sciences of the B.S.S.R., Academician of the Academy of Sciences of the B.S.S.R. V. F. Kuprevitch opened the conference. In his speech of welcome the President underlined that the problems of heat and mass transfer, which the conference was to deal with, were of paramount importance for the development of the most significant branches of modern science, and also for the solution of various engineering problems of further technical progress.

The President said: "The fact that such an important conference should be held in Minsk and in the Academy of Science of the B.S.S.R. is a great honour for us, a great tribute to the success achieved by the Institute of Energetics of the Academy of Science of the B.S.S.R., by Academician Alexei Vasilievich Luikov's school in the cause of the development of heat and mass transfer science."

The President expressed his belief that the conference would assist the strengthening of co-operation of Soviet scientists with scientists of other countries, would make a great contribution to the cause of consolidating friendship and co-operation among the peoples and to the cause of peace which was a goal of prime importance for every Soviet citizen.

The President let the Chairman of the Organizing Committee, Academician of the Academy of Sciences of the B.S.S.R., member of the Academy of Civil Engineering and Architecture of the U.S.S.R., Professor Alexei Vasilievitch Luikov have the floor.

In his address Academician A. V. Luikov said: "Processes of heat and mass transfer represent one of the most important branches of modern science and have a great practical significance for new engineering.

In connexion with the intensification of technical processes the transition to high regime parameters, a characteristic peculiarity of heat and mass transfer processes is revealed—their interrelation, i.e. heat and mass transfer of matter is one indivisible complex process. As an example, the classical case of heat transfer between a liquid and the surface of a solid may be given. While at low gas motion velocities and temperatures the heat transfer process may be considered as net transfer, at high velocities and temperatures followed by dissociation of gas molecules, the exchange of heat cannot be

described without taking into account the diffusion process, i.e. without the transfer of mass of the matter.

The so-called porous cooling of a wall exposed to a hypersonic stream of rarefied gas is a problem of extreme importance. In this case the process of heat transfer is solved as a complex heat and mass transfer problem. Generally speaking, when a solid has a capillary–porous surface structure then heat transfer is inseparably linked with mass transfer at any temperature and not only at low temperature and motion velocities of a liquid.

Heat transfer processes in the presence of phase conversions (boiling of liquids, fuel combustion, thermal decomposition, drying) represent by their physical nature one interconnected process of heat and mass transfer.

Interdependence of transfer of various substances takes place not only at heat and mass exchange, but also in the case of net heat transfer, such as combined convective and radiative heat transfer wherein the transfer of radiant energy is connected with the molar motion of a liquid.

Thus, at present, in most cases heat transfer is an indivisible interconnected process of heat and mass transfer. All this has determined both the scope and aims of our meeting, which is the first conference on heat and mass transfer. Sharing of experience on heat and mass transfer study in various thermo-energetic process and modern devices, choosing the most progressive methods of investigation and considering the possibility of applying results obtained in one branch of industry to other branches is the main task of our conference.

Whereas we have essential achievements in the field of engineering—in special sections of heat transfer by convection—(for example, investigations on heat and mass transfer in drying processes), on the other hand investigations on heat and mass transfer are still being carried out by obsolete methods. Suffice it to mention constructional heat engineering where heat and mass transfer investigation methods are still at the level of the beginning of the 20th century. Therefore active participation of specialists in constructional thermo-physics in meetings of this kind will be of great benefit to further development of constructional heat engineering. One can give many examples of unequal application of scientific achievements in heat and mass transfer theory to engineering researches. Now theoretical investigations are known to be of great importance. From this aspect we considered it reasonable to discuss the analytical methods of solution of heat and mass transfer differential equations, experimental methods of determining the thermo-physical properties of bodies under investigation as well as the application of similarity theory methods and thermodynamics of irreversible processes to heat and mass transfer study. Reports to be given at the plenary meetings will reflect the interconnection between the processes of heat and mass transfer, consider certain general methods of investigation, such as the application of thermodynamics of irreversible processes to heat and mass transfer research, as well as the modern state of the similarity theory or the method of generalized variables.

The task of our conference is to outline the main trends of heat and mass transfer progress; to work out proposals on urgent problems of theory, to consider and give recommendations on intensification of a number of technological processes.

The second and, perhaps, the principal task of the conference is to summarize the results obtained in the investigation of heat and mass transfer carried out in various fields of engineering. A number of these papers are published in almost inaccessible scientific editions and therefore it is extremely difficult for one to keep informed.

This conference is only the beginning of this important work which will be followed by the publication of all the reports and their discussions after the end of the conference. Such an encyclopedic publication on heat and mass transfer will be of great value for scientific workers, post-graduates, design engineers and its analysis will enable the thermo-physicist to choose the most reliable investigation method and the most accurate means of calculation.

Bearing this in mind, numerous reports on net heat transfer which are original both for the method of investigation and the results obtained will be heard at the conference.

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Leading Soviet scientists and specialists in the heat and mass transfer field will take part in the Conference. I believe those present here are quite familiar with their works. Well known scientists from other countries will also participate in our conference: Professor E. R. G. Eckert (the United States of America), Professor D. B. Spalding (Great Britain), Professor S. R. de Groot (the Netherlands), Professor J. P. Hartnett (the United States of America), Doctor C. Gazley (the United States of America), Professor Werner Jubitz (the German Democratic Republic), Professor U. Grigull (the German Federal Republic), Professor Zdzislaw Ziolkowski (the Polish Peoples' Republic), Professor Shandor Endreny and his colleagues (the Hungarian Peoples' Republic), Doctor A. J. Ede (Great Britain), Doctor Strach and his colleagues (C.S.S.R.).

We highly appreciate their kindness in consenting to make a number of interesting reports at our meeting.

In conclusion I would like on behalf of the Organizing Committee to wish all the participants of our conference fruitful work and active participation in discussions."

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The following reports were presented to the plenary session in accordance with the programme:

(1) P. K. KONAKOV, On mass and energy transfer, (U.S.S.R.).
(2) E. R. G. ECKERT, A. A. HAYDAY and W. J. MINKOWYCZ, Heat transfer, temperature recovery and skin friction on a flat plate with hydrogen release into a laminar boundary layer, (U.S.A.).
(3) J. P. HARTNETT and CARL GAZLEY, JR., A generalized presentation of mass transfer cooling for laminar flow over a flat plate, (U.S.A.).*
(4) S. S. KUTATELADZE, Heat transfer with boiling, (U.S.S.R.).
(5) D. B. SPALDING, Heat and mass transfer between the gaseous and liquid phases of a binary mixture, (Great Britain).
(6) S. R. DE GROOT, On the thermodynamics of irreversible heat and mass transfer, (Holland).
(7) A. V. LUIKOV, Application of methods of thermodynamics of irreversible processes to investigation of heat and mass transfer in a boundary layer, (U.S.S.R.).*
(8) L. E. KALIKHMAN, Problems of heat transfer in rarefied gases, (U.S.S.R.).

A great number of reports were presented at nine sectional sittings. The list of reports is given in Appendix I.

The work of the sectional sittings was summed up (Appendix II) and the resolution was adopted (Appendix III) at the final plenary sitting of the conference.

All the speakers noted the great importance of the conference for the further development of heat and mass transfer science

On behalf of the visitors from abroad, who were present at the conference, Professor E. R. G. Eckert (U.S.A.) said:

"Academician Luikov, members of the Organizing Committee of this conference, Colleagues, Ladies and Gentlemen.

I would like to use this opportunity to give in the name of the guests to this conference our sincere and heartfelt thanks for the opportunity to participate in the sessions of this meeting. We were all astonished at the large number of topics which were discussed, at the excellence of the papers and at the vigorous and thorough discussions. We would like to congratulate our Soviet colleagues on the excellent quality of the papers which were presented and which demonstrated the large volume of work in heat and mass transfer which is carried out in the Soviet Union.

We all hope that this meeting will be the beginning of an intensified and permanent co-operation of all scientists working in the field of heat and mass transfer regardless of the country in which they live. Thank you very much."

The Chairman of the Organizing Committee of the conference Academician A. V. Luikov closing the meeting said:

* These papers are not included in this special volume as they have already been published in *Int. J. Heat Mass Transfer* **3**, No. 3, 198–221 and 167–174 (1961).

"Dear Comrades and Gentlemen.

The scientific conference on problems of heat and mass transfer has come to an end. Its results are summarized at to-day's plenary session. The Organizing committee did its best to ensure a businesslike, creative and fruitful procedure and most favourable conditions for its work.

We note with satisfaction that according to the estimation of most delegates the procedures at the plenary and section meetings were successful. A great number of reports, information and speeches were made and discussed.

One conference, to be sure, cannot solve all the complicated scientific problems in the field of heat and mass transfer.

However, undoubtedly it made a valuable contribution to the development of this field of science, enabled discussion of some outstanding questions, summarized the experience available and outlined further trends of scientific research.

The excellent results of the conference can be accounted for not only by the work of the Organizing committee but mainly by the active participation of the delegates present.

I would like to express my gratitude to all those who delivered reports, gave information and made remarks in discussions, as well as to all commissions, interpreters, the secretariat and to all who promoted the successful procedure of the conference.

I wish to express high gratitude to our guests from other countries taking part in the conference and making for the strengthening of cooperation in science and engineering."

APPENDIX I

List of Reports made at Sectional Sittings of the Conference on Heat and Mass Transfer

Section I. Analytical methods of solution of heat and mass transfer problems

1. I. G. ALYAMOVSKY, The temperature field of a restricted body, having the form of a parallelopipedon, with a continuous heat source.
2. L. M. ALTSHULER, Analytical determination of the temperature field of a tube in a semi-infinite medium.
3. S. I. ANISIMOV and T. L. PERELMAN, Diffusion of charged particles in the presence of recombination.
4. G. P. BOIKOV, On problems of two-dimensional temperature fields.
5. O. S. BERLYAND, The method of solution of a certain heat conduction equation.
6. K. K. VOLKOVA, On a regular thermal regime in bodies with an arbitrary cylindrical form.
7. SH. L. GOLDIN, The stability of stationary regimes of a working medium motion in installations of cooling by evaporation.
8. S. N. DETKOV, On conformal reflections of radiation fields in vacuum.
9. E. I. KIM and B. B. BAIMUKHANOV, Boundary value solutions of the heat conduction equation with an explosive coefficient.
10. E. I. KIM and L. P. IVANOVA, A two-dimensional problem of heat and mass transfer with drying processes.
11. B. I. KOGAN and A. F. KHRUSTALEV, The temperature distribution in an infinite hollow cylinder.
12. V. I. MAKHOVIKOV, Several problems of thermal elasticity applying to a space having an infinite number of cylindrical orifices.
13. G. F. MUCHNIK, Solution of heat conduction problems by the "lattice" method.
14. T. L. PERELMAN, On conjugated problems of heat transfer.
15. G. V. PODDUBNYI, Application of coupled integral equations to the solution of a certain conduction problem.
16. A. A. POMERANTSEV, On the free convection theory.
17. I. G. PORTNOV, Solution of certain problems with phase conversions by the method of operational calculus.
18. L. A. ROTT and N. A. STODOLNIK, Nonlinear equation for diffusion near the stagnation point.
19. L. A. SVERGUNENKO, On the influence of heat conduction on the absorption of sound in crystals having defects.
20. L. M. SIMUNI, A numerical solution of certain problems of liquid motion with a variable viscosity.
21. A. G. TEMKIN, The temperature field of exothermic bodies.

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22. V. V. USANOV and G. V. TSIKLAURI, On the analytical determination of effective surfaces in channels in the presence of heat transfer and friction.
23. P. V. TSOI, Analytical solutions of a system of equations of heat and mass transfer for a semi-infinite medium with different boundary conditions.
24. P. V. CHERPAKOV, On the uniqueness of boundary value solutions in the theory of convective heat transfer.
25. V. L. SHEVELKOV, On the determination of the temperature field of materials under thermal treatment.
26. N. G. SHIMKO, The heat conduction problem for uniformly laminated cylinders with contact resistance.
27. P. P. YUSHKOV and L. I. LOGINOV, The numerical method of integration of one system of heat and mass transfer differential equations for the case of variable physical properties.

Section II. Calculation methods and modelling of heat and mass transfer processes

1. G. A. AKSELRUD, Transformation of heat and mass transfer processes in the system "porous body–liquid".
2. V. Y. BORODACHEV, L. M. KULIK and A. K. RUDKO, An approximate solution of the heat conduction equation for uniformly laminated media.
3. L. A. VULIS, I. F. ZHEREBYATIEV and A. T. LUKIYANOV, The solution of non-linear heat conduction equations on statistic electrical integrators.
4. A. N. GORDOV, Non-stationary temperature fields of solids under the conditions of changing heat transfer.
5. U. GRIGULL (G.F.R.), Certain problems of non-stationary heat conduction theory.
6. S. N. DETKOV, Resultant absorptivity and its calculation methods.
7. V. A. DENISOV, The method of solution of heat conduction problems in multilayer solids and its application to the solidification front advancement problem.
8. O. A. DOLGOV, The calculation of non-stationary heat transfer in rocks and freezing columns when shaft designing by a freezing method.
9. L. M. ZYSINA-MOLOZHEN and M. P. POLYAK, The calculation of the temperature field in the body of a cooled turbine blade.
10. A. V. KAVADEROV, Y. A. SAMOILOVICH and V. N. KALUGIN, Laws of heating bodies, having the simplest shapes, by radiation and convection.
11. A. P. KLIMENKO, G. E. KANAVETS, B. V. GAIDUK and E. I. CHERNOBYLSKAYA, The calculation method of the optimum heat exchangers on electron digital computers.
12. N. V. KLIMENTOV, The calculation method of hydrodynamic resistance and heat transfer for jet flowing of unrestricted barrier.
13. L. S. KLYACHKO, Heat and mass transfer both with free and forced convection.
14. L. A. KOZDOBA, Application of the electrical modelling method in ohmic resistance networks for solution of non-stationary thermal conductivity method.
15. L. A. KOZDOBA and V. I. MAKHNENKO, Electrical modelling of temperature fields for welding and soldering of objects having different shapes.
16. L. I. KUDRYASHEV and A. V. TEMNIKOV, The investigation of non-linear problems of non-stationary heat transfer by electrical modelling method.
17. L. I. KUDRYASHEV and V. K. LYAKHOV, Internal problem and heat transfer coefficient calculation method.
18. L. I. KUDRYASHEV and B. N. ASTRELIN, The effect of non-stationary state on heat transfer coefficient of solids of a spherical form in a flow with rather low Reynolds numbers.
19. L. I. KUDRYASHEV and R. N. KITOV, Determination of the universal heat transfer coefficient under the conditions of an external problem with regard to chemical conversions.
20. L. I. KUDRYASHEV and V. P. VESELOV, Investigation of non-stationary heat transfer processes in heat exchangers on electron models with regard to variability of thermal properties.
21. V. M. LYSYANSKY, On mass transfer in the process of extraction out of solid materials.
22. R. M. LADYZHENSKY, Calculation of injector air-coolers.

23. M. M. Litvinov, V. A. Kirillov and Oleshkevich, Semi-automatic electro-integrator for solving non-stationary heat conduction problems.
24. A. N. Lyapin, Turbulent mass transfer of a free stream as hydrodynamic factor determining its shape, kinematics and dynamics.
25. E. I. Molchanov, Application of the hydraulic analogy method to investigation of temperature fields in gas turbine elements.
26. I. R. Mikk, Emissivity of a cylindrical radiating volume.
27. V. I. Psarev, On calculation of heating and cooling kinetic curves.
28. G. I. Pavlovsky, The regularization of thermal processes in steam engines.
29. M. A. Pudovkin, A solution of the third linear heat problem with uniformly moving boundary in a semi-infinite area.
30. V. I. Rakhovsky, The investigation of non-stationary thermal regime of contacts with heavy current in vacuum.
31. Y. A. Samoilovich, The calculation of rectangular bodies heating to technological conditions.
32. V. S. Semenov, Electrical modelling of heat transfer processes in cylinder-piston group of internal combustion engines.
33. V. N. Timofeev, V. M. Malkin and F. P. Shklyar, The theory of regenerative heat exchanger calculation.
34. E. I. Taubman, On calculation method of heat transfer through a wall with an aggregation state change of one or both heat agents.
35. G. V. Filippov and A. I. Ku, Some applications of the Loitsyansky method of a boundary layer calculation under conditions of an internal problem.
36. G. N. Tretiyachenko and L. V. Kravchuk, Methods of "dangerous" temperature fields and evaluation of heat stability of objects made of fragile materials.
37. V. M. Steinberg, A new method of calculation of the non-stationary temperature field for a semi-infinite non-uniform complex of solids with thermal intercontact.
38. N. A. Yaryshev, Some problems of heat conduction theory of temperature sensitive elements used for non-stationary temperature measurements.
39. O. I. Yaroshevich, Application of hydraulic analogy method to the investigation of heat transfer processes in solids with internal heat sources.

Section III. General heat transfer problems (*No.* 1)

1. A. M. Asaturyan, V. A. Tonkoshkurov and V. I. Chernikin, On interaction of heat and hydrodynamic fields in a flow with variable viscosity.
2. L. Y. Artyukh, L. A. Vulis, V. P. Kashkarov and L. P. Yarin, Thermal problems of a boundary layer with heterogeneous and diffusive combustion.
3. K. A. Barlybaev, S. V. Bukhman, K. A. Zhurgenbaev and B. P. Ustimenko, Some problems of convective heat transfer in an incompressible liquid (internal problem).
4. V. P. Bakaleev, Fusion of a plate of finite thickness.
5. L. A. Vulis, On superimpressing of molecular and molar effects in the transient region of a flow.
6. L. A. Vulis, T. P. Leont'eva, I. B. Palatnik and Z. B. Sakipov, Thermal problems of a free-stream turbulent boundary layer.
7. I. P. Ginzburg, On possible solution methods of boundary layer problems with dissociation and diffusion.
8. I. P. Ginzburg, Z. S. Galanova and V. G. Dementiev, Solution of laminar boundary layer problems with regard to radiation and absorption of a medium.
9. A. S. Ginevsky and E. E. Soladkin, The effect of the diametrical curvature of a surface on heat transfer rate of axisymmetrical bodies and channels.
10. A. A. Gryaznov, On the theory of heat and mass transfer with convective motion of a liquid.
11. A. F. Gendelsman, On the determination of friction work with a gas flow in long tubes.
12. S. I. Gribkova and L. S. Shtemenko, The experimental investigation of slip and temperature jump with rarefied air flow near a solid wall.
13. N. N. Gvozdkov and E. P. Vaulin, On heat transfer of a porous plate in a gas flow.

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14. E. P. VAULIN, On heat and mass transfer with phase transition "solid–gas" on the surface.
15. A. N. DEVOINO, On certain results of the investigation of heat transfer by a rarefied gas with natural convection.
16. L. M. ZYSINA-MOLOZHEN, Determination of turbulent transfer constants in a compressible gas flow.
17. I. A. ZOTIKOV and L. I. BRONSKY, The experimental investigation of heat transfer with metal fusion and melted metal supply through a porous wall.
18. A. J. EDE (Great Britain), The heat transfer coefficient for flow in a pipe.
19. S. S. KUTATELADZE and A. I. LEONTIEV, Approximate methods of calculation both of heat transfer and friction with turbulent motion of a compressible gas.
20. L. S. KOKOREV, On turbulent diffusion of heat and momentum in a uniform flow.
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22. S. I. KOSTERIN and I. I. YUSHCHENKOVA, Structure and interaction of supersonic vapour streams in vacuum relative to types of vacuum pumps.
23. S. I. KOSTERIN and Y. A. KOSHMAROV, The investigation of flow and heat transfer in a flat supersonic nozzle for a rarefied gas applying to vacuum pumps.
24. Y. V. LAPIN, Mass transfer with turbulent flow of a compressible gas with foreign gas supply.
25. D. I. GYAKHOVSKY, On the distribution of turbulent pulsation rate intensity in streams.
26. S. K. MATVEEV, Determination of heat and mass transfer at the stagnation point of a blunt-nosed body at hypersonic speed.
27. A. A. POMERANTSEV, On the theory of fusion and scorching of a body (the Stephan problem).
28. B. S. PETUKHOV, Heat transfer and hydraulic resistance in tubes with turbulent flow of a liquid with variable physical properties.
29. P. N. ROMANENKO, A. I. LEONTIEV and A. N. OBLIVIN, The investigation of heat transfer and resistance with heated air motion in diffusers and mixers.
30. A. K. REBROV, Heat transfer of a cylinder with free motion of a gas in a rarefied space.
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32. G. A. TIRSKY, Fusion of a solid near the stagnation point in a flat and axisymmetrical gas flow.
33. M. M. FARZTDINOV, On the thickness of dynamic and temperature boundary layers with free convection in hollows.
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37. V. K. SHCHITNIKOV, Influence of body configuration on heat transfer in a forced air flow.
38. V. M. BORISHANSKY and E. D. FEDOROVITCH, Heat transfer of a plate in a flow with a wide range of Reynolds numbers.
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40. I. S. KOCHENOV, V. D. VILENSKY and Y. N. KUZNETSOV, Hydraulic resistance of non-stationary flows.
41. V. B. LEONAS, Experimental investigations of heat transfer in a free-molecular flow regime.

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2. V. A. BAUM, Influence of the mass transfer coefficient on water temperature distribution in the assembly of the water-moderated water-cooled reactor.
3. R. A. BAKHTIOSIN and Z. P. GORBIS, Experimental investigation on heat transfer of flows with dust particles.
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7. Z. R. GORBIS, On mechanism of heat transfer by two-phase flows "gas–solid particles".
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10. A. A. ZHUKAUSKAS, V. I. MAKARYAVICHYUS and A. A. SHANCHUYAUSKAS, On heat transfer of a bundle of smooth tubes in cross flow of a liquid.
11. S. S. ZABRODSKY, A fluidized bed as an intensification method of furnace gases cooling.
12. M. G. KAGANER and L. I. GLEBOVA, The effect of various factors on heat transfer through porous materials in vacuum.
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15. N. V. KRYLOV, Investigation of "cold bridges" in insulation constructions of deep cooling systems.
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17. L. I. KUDRYASHEV and S. V. SHCHIBRAEV, Heat transfer of bodies in a stream.
18. M. D. KUZNETSOV, Expression of experimental data through similarity numbers.
19. M. D. KUZNETSOV and P. O. NOVITSKY, On intensification of heat and mass transfer processes in a boiling layer.
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21. I. M. MASLENNIKOV, Experimental determination of heat radiation properties when heating bodies by radiation in a diathermic medium.
22. N. A. MOROZOV, Certain regularities of the cooling process under conditions of high temperatures and vibrating body motion.
23. V. P. MOTULEVICH, Y. N. PETROV and V. M. EROSHENKO, The effect of electrical fields on convective heat transfer.
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29. S. S. FILIMONOV and B. A. KHRUSTALEV, Local heat transfer and hydraulic resistance calculation with turbulent motion in tubes with various entrance conditions.
30. CHASHCHIN and N. N. NORKIN, Heat transfer of tubular surfaces with low fins.
31. I. T. SHVETS, E. P. DYBAN, M. V. STRADOMSKY and E. Y. EPIK, Experimental investigation of the flow turbulence influence on heat transfer of air motion in tubes.
32. V. K. SHCHERBAKOV, Peculiarities of heat transfer through a wall with longitudinal fins.
33. G. I. ELKIN and Z. R. GORBIS, On heat transfer in a quartz-sand–gas suspension dragged aerodynamically and mechanically.
34. V. JUBITZ (D.D.R.), Radiative heat transfer and its peculiarities.

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3. M. V. Buikov, The theory of thermal and diffusion relaxation of an evaporating drop.
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5. I. P. Vishnev, On the effect of vapour content on heat transfer of boiling in tubes.
6. L. N. Grigorovich, Heat transfer investigation with boiling of binary mixtures.
7. A. A. Komarovsky and V. V. Streltsov, Superposition of natural convection on forced one with mass transfer in a flow of liquid through an immovable granular layer.
8. A. P. Klimenko, G. E. Kanavets, B. V. Gaiduk and S. I. Chesnobylskaya, More accurate design of heat exchangers.
9. S. A. Kovalev, Certain laws of heat transfer with film boiling in a volume.
10. L. E. Mikhailov, Investigation of heat removal crisis for forced motion of ethyl alcohol in a circular channel.
11. P. A. Maslichenko, Data on film boiling mechanism in a great volume of liquid.
12. Z. L. Miropolsky, Stagnation heat flows with water boiling in channels.
13. I. L. Mostinsky, Application of the law of corresponding states to heat transfer calculation with boiling of a liquid.
14. M. E. Ivanov, On calculation variation of parameters of heat and mass transfer finite process both with condensation from "vapour–gas" mixture and evaporation into it.
15. P. A. Novikov, The effect of body motion rate on heat transfer and mass transfer in vacuum.
16. P. I. Povarnin, Application of the thermodynamic similarity method to calculation of a liquid surface tension.
17. P. I. Povarnin, Generalization of data on boiling crisis with temperature of a water flow in tubes below the saturation one.
18. N. A. Popov and V. I. Rakhovsky, On the impoverishment of metal ceramics by highly volatile components when heating in vacuum.
19. I. I. Paleev and F. A. Agafonova, Heat transfer between walls and a gas flow with drops of an evaporated liquid.
20. O. V. Remizov, Effect of the channel geometry on the critical heating with forced water flow.
21. K. I. Reznikovich, Heat and mass transfer on tubes with spiral fins.
22. N. U. Rizaev, Investigation of mass transfer process in "solid body–liquid" systems.
23. N. G. Styushin, New results on heat transfer investigation with surface boiling.
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25. G. T. Sergeev, Interinfluence of heat and mass transfer processes with evaporation.
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28. N. E. Hazanova, Diffusion in gases near the stagnation point of "liquid–vapour" equilibrium.
29. N. E. Hazanova, Diffusion in gases at high pressures.
30. K. P. Shumsky, On the theory of phase conversions in vacuum.
31. I. Schneller (Czechoslovakia), Some problems of heat and mass transfer studies in the National Research Institute of Heat Engineering in Prague.
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2. A. G. Belkin and Y. N. Suloev, On heat transfer problem in block polymerization processes.
3. G. V. Vasyunina and L. S. Akselrod, Determination of the useful life of air separator freezers.
4. R. G. Valdek and N. L. Lutskovskaya, On heat of organic substance decomposition in Estonian combustion shell.
5. N. I. Gelperin and V. B. Kvasha, Determination and maintenance of an optimum temperature field in chemical reactors.

6. G. N. DELYAGIN and B. V. KANTOROVICH, Mass transfer in the process of fuel combustion in a flow.
7. V. V. DILMAN and V. R. KUCHINSKY, Intensification of mass transfer in the cross flow of a gas and liquid.
8. V. I. DANILKIN and V. E. HARTSEV, Heat and mass transfer under non-isothermal conditions.
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10. F. F. ZIGMUND, On design of crystallizers with air cooling.
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12. B. V. KANTOROVICH, Heat and mass transfer in a fuel combustion process.
13. K. L. LEONOVICH, Aerodynamic means of intensifying heterogeneous processes.
14. I. L. LUBOSHITS, Deep drying of milling peat and its low-temperature carbonization in a falling bed with external heating.
15. V. P. MOTULEVICH, Convective heat transfer in the presence of non-equilibrium chemical reactors.
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18. R. S. PRASILOV, Dynamics of volatile ash deposition on screen heating surfaces of industrial furnaces.
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22. S. M. REPRINTSEVA, Thermal decomposition process of milling peat in a falling bed with external heating of a reactor.
23. A. V. RALKO, Heat and mass transfer at baking of clays and caolins.
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25. R. SH. SAFIN, N. M. ZHAVORONKOV and A. M. NIKOLAEV, The study of physical absorption and chemisorption processes in an apparatus of a rotational type.
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3. E. I. GUIGO and E. S. KOUKHCHSHVILI, Some theoretical principles of high efficiency sublimation condensator design.
4. V. I. ZHIDKO, Investigation of grain drying in a boiling layer.
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8. V. G. KARPENKO, Heat and mass transfer at drying and grinding of brown coal.
9. O. A. KREMNEV, V. R. BOROVSKY and A. A. DOLINSKY, Pulverizing evaporative-drying method of dehydrating solutions with high moisture content.
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14. G. A. MAKSIMOV, Hydroscopic properties of capillary-porous materials for various modes of drying.

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17. V. N. PRAVDIN and A. N. USPENSKY, Experimental investigation of heat and mass transfer of the ribbon-coating machine.
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20. B. S. SAZHIN, Loose material drying and development of aero-fountain dryer rational scheme.
21. P. T. SMENKOVSKAYA, External heat and mass transfer of food stuffs drying by sublimation in vacuum.
22. A. F. SOROKIN and Y. K. KOPTELEV, Radiative-convective pulsation dryer of belted cable MKSB isolation.
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2. Y. P. BARSKY, Stoneware baking process thermophysical parameters and the optimum baking temperature curve determination.
3. M. S. BELOPOLSKY, Optimum drying regime determination of plastic moulding stonewares.
4. E. Y. BRAININA and T. S. KARANFILOV, Investigation of heat and mass transfer in protecting the construction of buildings and in blocks by visualization and modelling.
5. M. A. BUZ, Heat and mass transfer of effective uranic stones at high coolant rates.
6. A. P. VASKOVSKY, Maintenance of definite temperature and humidity of air inside premises and stone walls in the Arctic.
7. K. P. VISHNEVSKY, Calculation of heat transfer in a granular material layer.
8. G. N. DANILOVA and S. N. BOGDANOV, Determination of thermophysical properties of certain concretes and gravels applied to concrete dam constructions.
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10. I. N. ZAIKA, Temperature control investigation of ceramics in constructions.
11. V. V. KOTELNIKOV, Investigation of building construction thermal engineering properties under scientific conditions.
12. O. A. KREMNEV, V. P. BOROVSKY and I. M. PIEVSKY, Investigation and experience of intensification of gypsum blocks and plank drying processes.
13. I. S. MELNIKOVA, Heat and mass transfer on protecting construction surfaces.
14. M. L. MIKHELSON, Kinetics of ventilation air heating at temperature below 0°C by latent heat release due to pulverized water freezing.
15. G. V. PODDUBNY, Heat transfer in the ground under the insulation of a refrigerator without a base.
16. M. Y. ROITMAN, On application of the nonstationary heat transfer theory to building construction refractivity calculation.
17. B. N. SREBNITSKY, Investigation of heat transfer in a hydroair heater with an irrigative nozzle.
18. K. S. STRELKOVA, Determination of temperature on the inner surface of panel joints by calculation methods.
19. S. L. FRID, Basic heat transfer problems arising from the design of large-scale hydrotechnical constructions.

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gation of their dependence on temperature and pressure.

2. K. I. Amirkhanov, A. P. Adamov and L. N. Levina, Thermal conductivity of carbon dioxide along the boundary curve including the area of critical state.
3. Y. P. Barsky, A non-stationary heat flow measurement and new methods of thermal properties investigation.
4. Y. P. Barsky, Thermal analysis as a method of investigation of heat and mass transfer with phase conversion.
5. L. A. Brovkin, On possible reasons for the effect of measured heat content increase of certain insulated bodies.
6. Y. D. Basilevskaya, Binary mixture diffusion coefficients of certain hydrocarbons and air.
7. A. V. Verzhinskaya, The method of constant energy source.
8. V. F. Vyshenskaya and N. D. Kosov, Investigation of temperature dependence of gas diffusion coefficient.
9. D. I. Gremilov, High temperature method of direct determination of heat transfer coefficient.
10. V. A. Gruzdev, The differential method of hydrostatic weighing to measure molten metal densities.
11. V. M. Gumenyuk, V. S. Ivanov and V. V. Lebedev, Determination of thermal conductivity of metals at temperature above 1000°C.
12. V. A. Grishin, Heat measurements by the method of current thermal compensation and some results of comparison.
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14. I. Y. Zalkind, A device determining heat flows from heated surfaces with temperatures up to 150°C.
15. Y. Zemanek (Czechoslovakia), Investigation of physical properties of liquids.
16. V. A. Kalendaryan, R. A. Bakhtiozin and Z. R. Gorbis, Thermophysical properties of synthetic graphite particles.
17. Y. A. Kirichenko, Determination of thermal coefficients by the method of radial temperature waves.
18. N. D. Kosov and O. V. Rivin, On a new type of calorimeter for thermal constant determination.
19. A. A. Konkov and V. P. Ionov, Spectral properties of certain gases at high temperatures.
20. M. V. Kulakov, A method and instruments for a complex determination of thermal coefficients.
21. E. V. Kudryavtsev and K. N. Chakalev, Standardization of heat flow.
22. R. E. Krzhizhanovsky, Influence of thermal treatment on thermal and electrical conductivities of heat resistant alloys.
23. E. R. Krzhizhanovsky, General laws in electric conductivity behaviour of metals and alloys.
24. I. S. Lisker, Non-stationary methods for determining thermal properties of semiconductors.
25. N. N. Medvedev, Temperature hysteresis.
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28. G. E. Ozhigov, Low inertia thermo-column for investigation of heat transfer by radiation.
29. B. P. Pashaev, Change of thermal conductivity of some metals and alloys on melting.
30. V. I. Petrovichev and L. S. Kokorev, A new non-stationary method of heat transfer coefficient measurement.
31. R. S. Prasolov, Materials with superlow heat conduction and calculation of gas thermal conductivity and density in fine-porous materials and rarefication.
32. E. S. Platunov, Apparatus for thermophysical tests developed at the Leningrad Institute of Precise Mechanics and Optics.
33. P. E. Suetin, Measurement of gas reciprocal diffusion coefficient by optical methods.
34. G. N. Tretyachenko and L. V. Kravchuk, High temperature thermophysical properties determination methods.
35. A. F. Chudnovsky, Determination method of semiconductor thermophysical properties.

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APPENDIX II

The Account of Sectional Procedure on Heat and Mass Transfer

The procedure of the Conference on Heat and Mass Transfer was carried out according to the programme. Eight hundred and thirty-five participants were organized in nine sections.

In their resolutions the participants emphasized both the great importance of the problems considered in the reports delivered and the high level of theoretical and experimental investigation.

The discussions were active and fruitful. Altogether three hundred and fifty participants took part in discussions at sectional sittings.

The sectional work was as follows:

Section I: Twenty-seven communications and reports were delivered. They were mostly devoted to problems on heat conduction, boundary layer theory, thermo-elasticity phenomena, numerical methods of the solution of problems.

Section II: Thirty-seven reports and communications were made which reflected the following principal problems:

(1) Development of calculation methods of heat distribution for non-stationary heat conduction;
(2) Development of electronic computing devices for the solution of heat and mass transfer problems;
(3) Improvement of methods of calculation of heat exchange apparatus.

Sections III and IV: Seventy-eight reports and communications were made where the following problems were considered:

(1) The boundary layer theory and its application to solving problems on heat and mass transfer in non-homogeneous media;
(2) Phenomenon of fusion and sublimation of bodies moving in supersonic flows;
(3) The theory of turbulent motion of a liquid;
(4) Heat transfer in rarefied gases and metals.

Section V: Thirty-seven reports and communications were delivered. The basic problems considered were as follows:

(1) Heat transfer with boiling in a wide range of pressures up to critical and in a region of high heat stresses;
(2) Heat and mass transfer with phase conversions;
(3) Certain problems on heat and mass transfer at evaporization.

Section VI: Twenty-three reports and communications were made. They dealt with problems on heat and mass transfer of fuel combustion in a flow, for intensification of heat and mass transfer processes with chemical conversions, thermal decomposition of solid fuel, etc.

Section VII: Twenty-four reports and communications were made. The basic problems considered were as follows:

(1) Application of the thermodynamic method to analysing and calculating drying processes;
(2) Investigation into physical properties of materials depending on the form of the bond of moisture with the material;
(3) Investigation into grain drying;
(4) Various drying methods: infra-red, convective–conductive, pulverization, sublimation, etc.

Section VIII: Twenty-one reports and communications were delivered. The basic problems considered were the following:

(1) Heat and mass transfer in protecting building constructions;
(2) Drying of building materials;
(3) Determination of thermophysical properties.

Section IX: Thirty-seven reports and communications were made where the following problems were considered:

(1) Determination methods of thermophysical properties of various materials and heat agents;
(2) Devices for determining thermophysical properties of materials, diffusion coefficient and other parameters.

The main suggestions proposed in the sections were reflected in the resolution of the Conference which was distributed among the participants of the Conference on Heat and Mass Transfer.

APPENDIX III

The Resolution Adopted at the Conference on Heat and Mass Transfer Held in Minsk, June 5–9, 1961

One of the basic trends of scientific and engineering progress which paves the way for the successful fulfillment of the Seven-Year Plan of growth of the National Economy of the U.S.S.R. is the creation of equipment which will prove to be perfect, highly productive, economical and reliable in operation. A great number of machines, apparatus and installations are working with the application of various technological processes involving heat and mass transfer. Creation and exploitation of this equipment are based on results of scientific achievements in the field of thermophysics.

The main problem in the modern science of heat and mass transfer is the continuation of the detailed study of problems which will reveal both the physical nature of the mechanism and laws of energy and matter transfer.

The development of such branches of science as thermodynamics of irreversible processes, the theory of turbulent motion of non-isothermal, non-uniform flows, the theory of non-equilibrium processes, the theory of motion of multiphase, multicomponent systems with discrete elements as well as the development of methods of determining and obtaining experimental data on physical properties of substances at extremely high temperatures, pressures and concentrations shows the great need to have these problems successfully solved.

Considerable successes achieved in the sphere of studying heat and mass transfer processes are acknowledged by the Conference. The great number of participants who attended this Conference and the considerable number of papers presented are indicative of this.

At the same time certain disconnection, insufficient co-ordination and exchange of investigation results, especially among the industrial establishments, is noted by the Conference.

As a result of discussions on reports and communications, the total number of which was three hundred, and in accordance with subsequent discussions the Conference considers that it is advisable to outline the following principal trends in the field of scientific investigation of heat and mass transfer:

(1) To continue the further development both of analytical and experimental methods of heat and mass transfer study on the basis of thermodynamics of irreversible processes; the theory of boundary layers; statistical physics; the theory of non-equilibrium processes and similarity theory.

(2) To extend a number of works dealing with the field of the theory of turbulent motion of non-isothermal, non-uniform flows; the theory of motion of multiphase and multicomponent systems with discrete elements; heat transfer in rarefied media and heat transfer intensification by intermediate liquids and fine-grained solid coolants.

(3) Taking into account the wide application of heat and mass transfer processes with phase conversions, the development of further investigation of boiling processes with high parameters and heat loads, those of evaporation under various conditions as well as with other phase and chemical conversions; and those of boiling of multicomponent solutions, should be considered necessary.

(4) To continue the further development of the theory in the field of heat and mass transfer in drying processes on the basis of the classical thermodynamics and thermodynamics of irreversible processes; as well as on the basis of analytical methods of solving problems on non-stationary heat and mass conduction; the investigation of bond forms of moisture with a material; and on a wide application of similarity theory.

(5) The development of new highly intensive drying methods ensuring a very high quality of materials as well as the clarification of optimal drying processes should be considered the most important problem in the field of theory and practice of drying.

The following is recommended:

(6) When investigating heat and mass transfer processes with chemical conversions to apply on a larger scale up-to-date methods

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of investigation including modelling of separate stages of a process and installations as a whole.

(7) When solving the problems of constructional thermophysics to utilize modern achievements in the field of heat and mass transfer more widely for production of building materials and in building construction.

(8) To develop new methods of determining thermophysical characteristics of various materials giving particular attention to obtaining data in the region of high temperature.

To overcome successfully the problems raised the Conference considers that it is necessary to carry out the following practical measures:

(1) To call regular meetings and conferences both on general problems of heat and mass transfer and on separate problems with invitation of foreign scientists to such meetings;

(2) To organize a co-ordinational centre bound up with the Scientific Research Co-ordination Committee, attached to the Council of Ministers of the U.S.S.R. to co-ordinate all the investigations into heat and mass transfer;

(3) To apply to the Ministry of Higher and Secondary Special Education of the U.S.S.R. to consider the question of organization of heat and mass transfer laboratories in leading Institutes of Higher Education in our country;

(4) To apply to the Ministry of Higher and Secondary Special Education of the U.S.S.R. to make provision in curriculums for some thermotechnical specialization, the substitution of heat conduction courses for the heat and mass transfer course and to announce a competition for new textbooks for this course for main specialities;

(5) To apply to the Minister of Higher and Secondary Special Education of the U.S.S.R. to establish chairs of constructional thermophysics in Institutes of Civil Engineering;

(6) To apply to the Ministry of Higher and Secondary Special Education of the U.S.S.R. to establish additional financial support for the Journal of Engineering Physics and to change it into the organ of the Academy of Science of the B.S.S.R. and of the Ministry of Higher and Secondary Special Education of the U.S.S.R. To consider whether it is advisable to rename the Journal of Engineering Physics as the Journal of Engineering Thermophysics;

(7) To apply to the State Energetics Publishing House to increase the number of publications on heat and mass transfer problems.

(8) The Conference considers that it is necessary to carry out the work required to establish a standard system of terminology and symbols used in the field of heat and mass transfer;

(9) To apply to the Academy of Science of the B.S.S.R. to publish the proceedings of the present Conference.

The Conference highly appreciates the enormous preliminary work carried out by the Institute of Energetics of the Academy of Science of the B.S.S.R., by the Organization Committee and by the Chairman of the Organization Committee of the Conference, Academician A. V. Luikov.

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HEAT TRANSFER, TEMPERATURE RECOVERY AND SKIN FRICTION ON A FLAT PLATE WITH HYDROGEN RELEASE INTO A LAMINAR BOUNDARY LAYER

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Abstract—An analysis is presented for a two-component laminar boundary layer on a surface with zero pressure gradient and with mass release. The system of partial differential equations describing the velocity, temperature, and concentration field is converted by a similarity transformation into a system of integral equations. Numerical solutions are obtained on electronic digital computers for air flow with hydrogen injection, for Mach numbers 0 to 12, for free-stream temperatures 123 to 2000°R, and for wall-to-free stream temperature ratios from $\frac{1}{2}$ to 6. Solutions for the condition of zero temperature gradient in the boundary layer at and normal to the wall surface lead to temperature recovery factors. The results are utilized to investigate the validity and accuracy of engineering correlations presented in [15]. The proposed correlations describe the reduction in heat flux into the wall surface and in skin friction reasonably well. No simple correlations have been found for the dependence of hydrogen mass fraction at the surface on the mass release rate and for the temperature recovery factor. Values for a free-stream temperature of 123°R show generally the largest deviations from the rest of the data.

NOMENCLATURE

a, sound velocity;
c_f, local skin friction coefficient;
c_p, specific heat at constant pressure;
f, dimensionless stream function;
h, enthalpy;
k, thermal conductivity;
$\dot{m}$, mass flow per unit area and time;
q, heat flow per unit area and time;
r, recovery factor;
t, temperature;
u, velocity component parallel to surface;
v, velocity component normal to surface;
x, co-ordinate along the body;
y, co-ordinate normal to the body;
D, diffusion coefficient;
M_1, molecular weight of coolant gas;
M_2, molecular weight of pure air;
T, temperature ratio;
W, mass fraction of foreign gas.

Greek symbols

α, thermal diffusivity;
γ, ratio of specific heats;
η, dimensionless wall distance;
μ, dynamic viscosity;
ν, kinematic viscosity;
ρ, density;
φ_c, normalized mixture thermal capacity $(c_p/c_{p\infty})$;
φ_{c12}, normalized pure-component thermal capacity difference $(c_{p1} - c_{p2})/c_{p\infty}$;
φ_k, normalized mixture thermal conductivity (k/k_∞);
φ_μ, normalized mixture viscosity (μ/μ_∞);
φ_ρ, normalized mixture density (ρ/ρ_∞);
ψ, stream function.

Dimensionless parameters

c_f, local skin friction coefficient, $\left(\mu \frac{\partial u}{\partial y}\right)_w \Big/ (\frac{1}{2}\rho_\infty u_\infty^2)$;
Le, Lewis number, α/D;
M, Mach number, u/a;
Pr_∞, Prandtl number, $\mu_\infty c_{p_\infty}/k_\infty$;
r, temperature recovery factor, $(t_r - t_\infty)/(t_t - t_\infty)$;
$Re_{x,\infty}$, Reynolds number, $\rho_\infty u_\infty x/\mu_\infty$;
$Re_x^{*\prime}$, Reynolds number, $\rho^{*\prime} u_\infty x/\mu^{*\prime}$.

Subscripts

1, refers to pure coolant;
2, refers to pure air;

w, evaluated at wall conditions;
i, i-th component;
R, refers to radiation;
r, refers to recovery conditions;
∞, outside the boundary layer;
0, at solid surface;
s, refers to solid wall;
t, refers to total state.

Superscripts

$*'$, evaluated at the reference temperature, equation (11).

1. INTRODUCTION

RECENT advances in high speed and space flight technology have focused attention on the aerodynamic heating problem. The search for effective cooling methods for the protection of re-entering vehicles against the intense heat generated by friction in the atmosphere has led to the development of "mass-transfer cooling". This name includes all methods in which a mass flow of coolant gas is created from the cooled surface into the hot boundary layer irrespective of the particular way in which this mass is released. In transpiration cooling, for example, the coolant gas is ejected through a porous wall into the boundary layer. In ablation cooling, the coolant gas is produced at the surface by sublimation or some chemical reaction of the wall material itself. In liquid film cooling, evaporation of a liquid from the surface of a film covering the cooled wall produces the coolant gas. The term mass-transfer cooling focuses attention on the conditions in the boundary layer itself and on the way they are influenced by the mass release at the surface.

An analysis of mass-transfer cooling thus has to deal with boundary layer flow of a more component—at least two component—gas mixture composed of the fluid in the main stream and of the coolant and has to investigate the momentum, energy, and mass transport within this layer. Such analyses were at first carried out with the assumption that the fluid properties can be considered as constant within the boundary layer [1, 2]. The results indicated that heat transfer to the wall surface can be considerably reduced by the "blocking effect" which the coolant gas produced by moving away from the wall surface. Subsequent calculations, which considered the variations of the physical properties with composition and temperature, indicated that gases with low molecular weight are especially effective as coolant, insofar as they produce the largest decrease of the heat flow into the wall surface at a fixed coolant mass release [3, 4, 5, 7]. Hydrogen, as the gas with the smallest molecular weight, appears, therefore, especially promising.

There are two types of flow which are well suited for an analysis: plane or rotationally symmetric stagnation flow and flat plate flow. Results obtained for these are valuable since they can be considered as limiting cases of the behavior of boundary layers along the surface of objects exposed to a subsonic or supersonic flow. Analyses for laminar boundary-layer flow over a surface with constant pressure (flat plate) and with hydrogen as coolant are reported in [5, 7]. Laminar stagnation-point flow has been treated in [8]. Numerical calculations in these reports have been made for a temperature outside the boundary layer which is close to 400°R. In the present paper, the calculations are extended to a wide range of boundary conditions including free-stream temperatures up to 2000°R, keeping in mind that the results of a "flat plate" analysis should also describe, with good approximation, conditions for boundary layers on slender objects or on blunt objects in regions where the pressure gradients are small. In such cases, the air outside the boundary layer has passed at some upstream location through a shock and has been heated to a high temperature, especially in hypersonic flow. The results of the analysis are used to investigate the validity and accuracy of engineering correlations which have been proposed in [15] and which are in widespread use today.

It is assumed in this paper that no dissociation or chemical reactions occur within the boundary layer. In reality, such reactions will always be present at the higher temperature levels. Their inclusion in the analysis, however, is hampered by insufficient knowledge of chemical reaction kinetics and of the transport properties involved and has been included in boundary layer analyses only for gas mixtures with a Lewis number equal or close to one, for example in [9]. This condition

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is, unfortunately, not too well fulfilled for hydrogen–air mixtures. Therefore, only approximate estimates on the effect of reactions on the heat transfer in such a boundary layer can be made at present and more exact analysis has to await the accumulation of more knowledge on the above-mentioned parameters.

Only a few examples of the results of this analysis can be presented in this paper. Complete information will be contained in a later publication.

2. ANALYTICAL FORMULATION

Basic equations

The equations for the steady, laminar, binary mixture boundary layer on a surface with constant pressure in flow direction (flat plate) may be written as follows:

Continuity:

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0; \qquad \text{(I)}$$

Momentum:

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} - \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) = 0; \qquad \text{(II)}$$

Diffusion:

$$\rho u \frac{\partial W}{\partial x} + \rho v \frac{\partial W}{\partial y} - \frac{\partial}{\partial y}\left(\rho D_{12} \frac{\partial W}{\partial y}\right) = 0; \qquad \text{(III)}$$

Energy:

$$\rho c_p u \frac{\partial t}{\partial x} + \rho c_p v \frac{\partial t}{\partial y} - \frac{\partial}{\partial y}\left(k \frac{\partial t}{\partial y}\right) - \mu \left(\frac{\partial u}{\partial y}\right)^2 + \rho D_{12}(c_{p2} - c_{p1}) \frac{\partial t}{\partial y} \frac{\partial W}{\partial y} = 0. \qquad \text{(IV)}$$

These equations are based, apart from the standard boundary layer arguments, on the assumption that the mixture is inert and that thermal diffusion is negligible. The physical properties which enter into the equation system (I) to (IV) are assumed to be given functions of mass concentration W and temperature t. The specific formulas are discussed in detail in the section on Physical Properties. All symbols are defined in the List of Nomenclature.

The energy equation has been written in various recent publications in enthalpy as dependent parameter rather than in temperature. This simplifies the equation considerably for a gas mixture with a Lewis number equal to one. It offers, however, no advantage for a gas mixture with a different Lewis number. Therefore, equation (IV) is based on temperature.

W denotes the mass fraction of coolant gas (hydrogen) released at the surface. In the following numerical calculations, the mainstream will be assumed to be air and the mass fraction of coolant outside the boundary to be zero.

Two sets of boundary conditions are considered. They differ only in the boundary conditions imposed on the energy equation, equation (IV), for which either the wall temperature or a temperature gradient zero in the fluid at the wall is prescribed. Specifically, we state the following:

for $y = 0$: $u = 0$

$v = v_w(x)$

$t = t_w$ or $\left(\frac{\partial t}{\partial y}\right)_{y=0} = 0$

for $y \to \infty$: $\lim u(x,y) = u_\infty = \text{const.}$

$\lim W(x,y) = 0$

$\lim t(x,y) = t_\infty = \text{const.}$

Whenever $t = t_w$ at $y = 0$ is imposed, we shall speak of the cooled wall case. If $(\partial t/\partial y)_{y=0} = 0$ is imposed, we shall term this the recovery situation.

On the assumption that at the surface the net mass flow of air is zero, we obtain

$$(1 - W_w)\rho_w v_w = -\rho_w D_{12w}\left(\frac{\partial W}{\partial y}\right)_w. \qquad (1)$$

Thus, either W_w or v_w may be prescribed independently, the other being determined by the above side condition.

With the system of equations (I) to (IV), the boundary conditions, and equation (1) the problem is completely defined. This implies that the way that the coolant mass flow leaving the wall surface is created (by transpiration, ablation, evaporation) has no influence on the boundary layer development as long as the wall temperature, t_w, and the injection parameter, v_w, are

prescribed. For a better understanding of the parameters in which the results of the analysis are presented, however, it is useful to consider, at this time, a general energy balance on the interface between the wall and the fluid, as shown in Fig. 1. The wall may be a porous structure

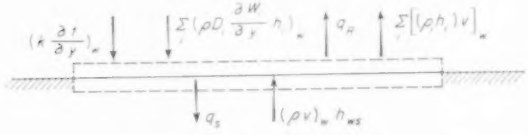

FIG. 1.

through which the coolant gas is blown, or it may be a solid at the surface of which the coolant gas is generated by sublimation.

The net heat input, q_s, to the solid wall material, is given for an i-component gas mixture by the relation:

$$q_s = \left(k\frac{\partial t}{\partial y}\right)_w + \left[\sum_i \rho\, D_i \frac{\partial W_i}{\partial y} h_i\right]_w + (\rho v)_w h_{ws} - [\Sigma_i (\rho_i h_i) v]_w - q_R.$$

The above expression states that energy is transported from the gas mixture to the interface by conduction $[k(\partial t/\partial y)]$ and by diffusion $[\Sigma_i \rho D_i (\partial W_i/\partial y) h_i]$. Simultaneously, there occurs an energy flux from the inside of the wall towards the interface at the rate $(\rho v)_w h_{ws}$. In transpiration cooling this energy flux is transported by the coolant gas as it moves toward the wall surface. In ablation cooling $(\rho v)_w$ is the rate with which the interphase recedes into the solid wall. The enthalpy h_{ws} is equal to the enthalpy of the coolant gas, as it passes the surface, for transpiration cooling, and equal to the enthalpy of the solid or liquid in ablation or evaporation cooling. Finally, energy is transported away from the interface by convection $[\Sigma_i (\rho_i h_i) v]$ and radiation (q_R). q_s is the heat conducted away from the surface in the solid wall material. For the binary system at hand, the above equation reads

$$q_s = \left(k\frac{\partial t}{\partial y}\right)_w + \left[\rho\, D_{12}\frac{\partial W}{\partial y}(h_1 - h_2)\right]_w + (\rho v)_w h_{ws} - [(\rho_1 h_1 + \rho_2 h_2) v]_w - q_R.$$

The total mass fluxes of hydrogen (1) and air (2) are given by

$$\dot{m}_{1w} = -\left(\rho\, D_{12}\frac{\partial W}{\partial y}\right)_w + \rho_{1w} v_w$$

and

$$\dot{m}_{2w} = \left(\rho\, D_{12}\frac{\partial W}{\partial y}\right)_w + \rho_{2w} v_w = 0.$$

Hence

$$\dot{m}_w = \dot{m}_{1w} + \dot{m}_{2w} = (\rho v)_w = \rho_{1w} v_w - \left(\rho\, D_{12}\frac{\partial W}{\partial y}\right)_w.$$

Substitution of these expressions into the above equation yields

$$q_s = \left(k\frac{\partial t}{\partial y}\right)_w - (\rho v)_w (h_{1w} - h_{ws}) - q_R. \quad (2)$$

The second term on the right-hand side vanishes for transpiration cooling. The first term on the right-hand side of equation (2) is determined by the boundary layer equations and their boundary conditions. It will be denoted by

$$q = -\left(k\frac{\partial t}{\partial y}\right)_w \quad (3)$$

and is presented in the later part of this paper. The rest of the terms in equation (2) depend on the cooling method, on the radiative exchange process, and on the temperature history of the wall and has to be calculated from considerations outside the scope of this paper.

In the recovery problem, by definition,

$$q = -\left(k\frac{\partial t}{\partial y}\right)_w = 0.$$

A recovery factor r is also defined by

$$r = \frac{t_r - t_\infty}{t_t - t_\infty} = \frac{T_r - 1}{T_t - 1}. \quad (4)$$

The recovery factor is a parameter which was found useful in the analysis of heat transfer

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from a high velocity gas to a solid surface, and in transpiration cooling in a constant property boundary layer [1].

Similarity transformation

We seek a class of "similar" solutions. Introduction of the stream function ψ defined by

$$u = \frac{\rho_\infty}{\rho}\frac{\partial \psi}{\partial y}$$

and

$$v = -\frac{\rho_\infty}{\rho}\frac{\partial \psi}{\partial x}$$

satisfies the continuity equation identically. The similarity variable

$$\eta = \tfrac{1}{2}\sqrt{\left(\frac{u_\infty}{\nu_\infty x}\right)}\int_0^y \frac{1}{\varphi_\mu}\,\mathrm{d}y \qquad (5)$$

and the assumption

$$\psi = \sqrt{(u_\infty/\nu_\infty x)}\, f(\eta)$$

transforms the equations (II) to (IV) into the following system:

Momentum:

$$\left[\left(\frac{f'}{\varphi_\mu \varphi_\rho}\right)'\right]' + f\left(\frac{f'}{\varphi_\mu \varphi_\rho}\right)' = 0; \qquad (\mathrm{II}')$$

Diffusion:

$$\left(\frac{W'}{Sc}\right)' + fW' = 0; \qquad (\mathrm{III}')$$

Energy:

$$\frac{1}{Pr_\infty}\left(\frac{\varphi_k T'}{\varphi_\mu}\right)' + \varphi_{c_{12}}\, Sc\, W'\, T' + \varphi_c f T' + \tfrac{1}{4}(\gamma_\infty - 1)\, M_\infty^2 \left[\left(\frac{f'}{\varphi_\mu \varphi_\rho}\right)'\right]^2 = 0. \qquad (\mathrm{IV}')$$

The properties have been made dimensionless by referring them to the respective property outside the boundary layer. The temperature ratio $T = t/t_\infty$ has been introduced. The primes denote differentiation with respect to η. The equation system (II′), (III′), (IV′), may also be thought of as being obtained from (I) to (IV) by a double transformation. One first introduces the Blasius variable

$$\eta_B = \tfrac{1}{2} y \sqrt{\left(\frac{u_\infty}{\nu_\infty x}\right)}$$

and transforms the resulting set of equations in η_B a second time by introducing

$$\eta = \int_0^{\eta_B} \frac{1}{\varphi_\mu}\,\mathrm{d}\eta_B.$$

The result, with the proper reinterpretation of $f(\eta)$, is again the equation system (II′) to (IV′).

The boundary conditions become:

for $\eta = 0$: $f' = 0$,

$f = f_w = \text{const.}$,

$W = W_w$,

$T = T_w$ or $T' = 0$,

for $\eta \to \infty$: $\lim f' = 2$,

$\lim W = 0$,

$\lim T = 1$.

The side condition, equation (1), transforms to

$$f_w = \frac{W_w'}{Sc_w\,(1 - W_w)}. \qquad (7)$$

Again it should be noted that either f_w or W_w may be prescribed independently, but not both.

Equations (II′) to (IV′) comprise a set of nonlinear ordinary differential equations which must be solved simultaneously satisfying one of the sets of boundary conditions given above. The interdependence between (II′), (III′) and (IV′), is strongly affected by the particular choice of property relations. For example, if the properties were assumed constant, (II′) could be solved independently of (III′) and (IV′). The solution of (II′) would then be used to solve (III′) and (IV′). Such procedure is evidently especially simple. For this reason, one seeks forms of the similarity variable η which give, in the final working equations (II″) to (IV″), combinations of properties as little dependent of local mass concentration and temperature as possible. This appears to be accomplished with the choice of η used in this paper at

least for light-weight gases such as hydrogen or helium. In fact, the energy equations in the integral form (equations IV'' and IV''') are in this way effectively uncoupled from the remaining equations since the property combinations are only mild functions of temperature. The sets of equations thus tend to behave as if the momentum and diffusion equations formed a simultaneous system with an independent energy equation.

3. METHOD OF SOLUTION

The equation system (II') to (IV') may be written as a set of integral equations. For the cooled wall one obtains:

Momentum:

$$f'(\eta) = C_1 \varphi_\mu \varphi_\rho \int_0^\eta \exp\left(-\int_0^{\eta'} f \, d\eta''\right) d\eta', \quad \text{(II'')}$$

$$f(\eta) = f_w + \int_0^\eta f' \, d\eta';$$

Diffusion:

$$W(\eta) = C_2 \int_0^\eta Sc \exp\left(-\int_0^{\eta'} Sc\, f \, d\eta''\right) d\eta'; \quad \text{(III'')}$$

Energy:

$$T(\eta) = T_w + C_3 \int_0^\eta \frac{\varphi_\mu}{\varphi_k} \exp\left(-\int_0^{\eta'} \lambda \, d\eta''\right) d\eta'$$

$$- \int_0^\eta \frac{\varphi_\mu}{\varphi_k} \exp\left(-\int_0^{\eta'} \lambda \, d\eta''\right)$$

$$\left[\int_0^{\eta'} g \exp\left(\int_0^{\eta''} \lambda \, d\eta'''\right) d\eta''\right] d\eta'. \quad \text{(IV'')}$$

In these equations, $\lambda(\eta)$ and $g(\eta)$ are defined as

$$\lambda(\eta) = Pr_\infty \frac{\varphi_\mu}{\varphi_k} \left(\varphi_c f + \frac{\varphi_{c12}}{Sc} W'\right)$$

and

$$g(\eta) = Pr_\infty \left\{ \tfrac{1}{4} (\gamma_\infty - 1) M_\infty^2 \left[\left(\frac{f'}{\varphi_\mu \varphi_\rho}\right)'\right]^2 \right\}.$$

The constants C_1, C_2, C_3 are obtained from the conditions at infinity:

$$C_1 = \frac{2}{\int_0^\infty \exp\left(-\int_0^{\eta'} f \, d\eta''\right) d\eta'},$$

$$C_2 = \frac{-W_w}{\int_0^\infty Sc \exp\left(-\int_0^{\eta'} Sc\, f \, d\eta''\right) d\eta'},$$

$$C_3 = \frac{\{1 - T_w + \int_0^\infty \frac{\varphi_\mu}{\varphi_k} \exp\left(-\int_0^{\eta'} \lambda \, d\eta''\right) \left[\int_0^{\eta'} g \exp\left(\int_0^{\eta''} \lambda \, d\eta'''\right) d\eta''\right] d\eta'\}}{\int_0^\infty \frac{\varphi_\mu}{\varphi_k} \exp\left(-\int_0^{\eta'} \lambda \, d\eta''\right) d\eta'}.$$

If the conductive heat flux at the surface is zero, equations (II''), (III'') remain unchanged. The energy equation (IV'') is replaced by

$$T = T_w - \int_0^\eta \frac{\varphi_\mu}{\varphi_k} \exp\left(-\int_0^{\eta'} \lambda \, d\eta''\right)$$

$$\left[\int_0^{\eta'} g \exp\left(\int_0^{\eta''} \lambda \, d\eta'''\right) d\eta''\right] d\eta' \quad \text{(IV''')}$$

where

$$T_w = 1 + \int_0^\infty \frac{\varphi_\mu}{\varphi_k} \exp\left(-\int_0^{\eta'} \lambda \, d\eta''\right)$$

$$\left[\int_0^{\eta'} g \exp\left(\int_0^{\eta'} \lambda \, d\eta'''\right) d\eta''\right] d\eta'.$$

The equation systems (II'') to (IV'') or (II''), (III'') and (IV''') are now solved by successive approximations.

For cooled wall calculations, a straightforward iterative technique is satisfactory. The procedure is the following. Let the boundary layer profiles W_0, T_0, f_0' and the blowing parameter f_{w0} be the initial approximations. Substituting these into the right-hand side of equations (II''), (III''), (IV'') and utilizing equation (7) yields the next set of values W_1, T_1, f_1', f_{w1}. This cycle is repeated until

$$\left.\begin{aligned} &\left|\int_0^\infty (W_{i+1} - W_i) \, d\eta\right| < \delta \\ &\left|\int_0^\infty (T_{i+1} - T_i) \, d\eta\right| < \delta \\ &\left|\int_0^\infty (f_{i+1}' - f_i') \, d\eta\right| < \delta \\ &\left|\int_0^\infty (f_{wi+1} - f_{wi}) \, d\eta\right| < \delta \end{aligned}\right\} \quad (8)$$

where δ was arbitrarily taken as 10^{-4}. The subscript i denotes here the i-th iteration. It may be shown that the uniform convergence of the above integrals constitutes a sufficient condition for the convergence of the integral equations.

In the recovery problem, the solution method is modified as follows. Let W_0, T_0, f_0', f_{w0}, be the set of initial values. Then the equations (II''), (III''), (IV'''), together with equation (7), yield

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the set of values W_1, T_1, f_1', f_{w1}. We consider W_1, f_1', f_{w1} to be the next approximations. The new approximation to T, however, is not T_1, but a certain perturbed set of values, $T_1^{\circledast}$, given by

$$T_1^{\circledast} = T_0 + \sigma(T_1 - T_0).$$

The function σ was taken as a positive constant $\sigma = 0{\cdot}3$ for all calculations. This procedure yielded rapid convergence even for "very bad" initial guesses. Without this modification, the iteration method exhibits occasionally some instability. When it occurs, depends on the deviation of the initial guesses from the final results.

The numerical calculations were performed on a Remington Rand 1103 digital electronic computer.

4. PHYSICAL PROPERTIES

Detailed derivations of the various relationships determining transport properties of multicomponent gas mixtures as functions of mass concentration and temperature are described in [10, 11]. In this paper, we shall merely state the main assumptions regarding the properties appearing in the previous equations and summarize the working formulas. The subscript 1 in these relations refers to the injected hydrogen and subscript 2 to air.

(1) *Mixture Schmidt number*, $Sc = \nu/D_{12}$, is assumed dependent on T only and is calculated according to [10].

(2) *Pure component thermal capacities*, c_{p1} and c_{p2}, were selected from [12, 13].

(3) *Mixture viscosity ratio*, φ_μ, is calculated by Wilke's method [11] with the following formula:

$$\varphi_\mu = \frac{\mu_1/\mu_\infty}{1 + 0{\cdot}1297\left(\dfrac{1-W}{W}\right)} + \frac{\mu_2/\mu_\infty}{1 + 3{\cdot}975\left(\dfrac{W}{1-W}\right)}.$$

The pure component viscosities, μ_1 and μ_2, are taken from NBS-NACA tabulation, [13]. Some extrapolation to higher temperatures was necessary in a few cases.

(4) *Mixture conductivity ratio*, φ_k, is calculated by Enskog's method, [10, 11] as follows:

$$\varphi_k = \frac{k_1/a_1 + k_2/a_2 + 0{\cdot}289\sqrt{(k_1k_2)/a_1a_2}}{\left(1 - \dfrac{0{\cdot}0209}{a_1a_2}\right)k_\infty}$$

where

$$a_1 = 1 + 0{\cdot}1886\left(\frac{1-W}{W}\right)$$

$$a_2 = 1 + 14{\cdot}228\left(\frac{W}{1-W}\right).$$

The pure component conductivities, k_1 and k_2, are calculated by Hirschfelder's method [10].

(5) *Mixture density ratio*, φ_ρ, is calculated assuming perfect-gas behavior for each component as

$$\varphi_\rho = \frac{t_\infty}{(1 + 13{\cdot}5\,W)t}.$$

(6) *Mixture thermal capacity ratio*, φ_c, is calculated for perfect gases as

$$\varphi_c = \frac{c_{p1}}{c_{p\infty}}W + \frac{c_{p2}}{c_{p\infty}}(1 - W).$$

The properties used in the present analysis do not include the effect of dissociation and chemical reactions. This had been done, because such effects are excluded from the whole analysis and since it is actually easier to generalize the results to a situation in which chemical reactions occur when transport properties excluding these effects have been used. More will be said about this point in the section on engineering correlations.

5. RESULTS

The following range of the parameters on which the calculations depend has been selected:

$$0 \leqslant W_w \leqslant 0{\cdot}8, \quad 0{\cdot}5 \leqslant T_w \leqslant 6, \quad 0 \leqslant M_\infty \leqslant 12,$$
$$123^\circ \leqslant t_\infty \leqslant 2000^\circ\text{R}.$$

The low free-stream temperatures are indicative of high speed wind-tunnel tests. It is believed that the results of the present analysis provide also a good approximation to laminar boundary layers on flying objects in regions with small pressure gradient. On such objects, however, the

air outside the boundary layer has passed a shock and is, therefore, heated to considerable temperatures. For this reason, the higher values of t_∞ have been included.

Some of the skin friction results are presented in Figs. 2 and 3. The subscript zero refers to a pure air boundary layer on a solid wall ($W_w = 0$).

The coefficient c_f is defined in the standard way as

$$c_f = \frac{\left(\mu \frac{\partial u}{\partial y}\right)_w}{\frac{1}{2}\rho_\infty u_\infty^2}.$$

One observes a marked reduction of the skin

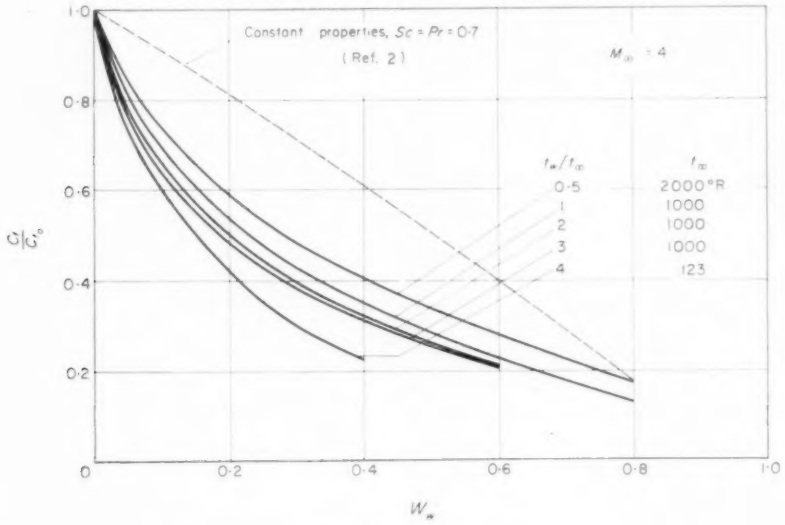

FIG. 2. Hydrogen–air, T_w = const.

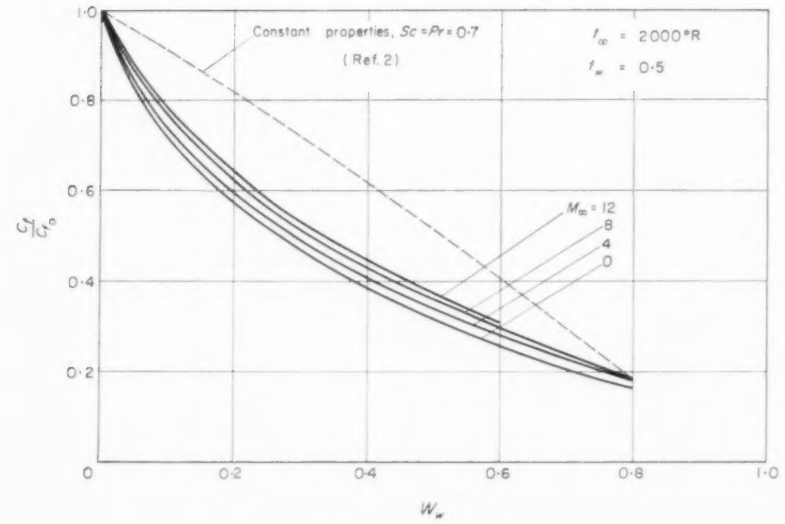

FIG. 3. Hydrogen–air, T_w = const.

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friction even for small hydrogen mass fraction at the wall. The reduction is larger for low values of the free-stream temperature and large values of the wall temperature, whereas the influence of Mach number is comparatively small. The dashed lines present the results of a calculation which assumes the properties to be constant and to have the same value for the main-stream fluid and for the coolant. Such calculations will approximate, for instance, situations with small Mach number, small differences between wall and free-stream temperature and with a coolant gas with properties not too different from those of the main-stream fluid. It is interesting to note that the reduction of the friction factor is, in this case, considerably smaller, especially at small values of the wall concentration.

Figures 4 and 5 present corresponding information on heat transfer to the surface. The heat flux, q, is defined in equation (3). It will be noted that the dependency of the heat-flux parameter, q/q_0, on the various boundary conditions is qualitatively the same as the one of the friction factor. Quantitatively, the parameters have a somewhat stronger influence on heat transfer than on wall friction.

Figure 6 presents the relation between the mass release parameter, f_w, and the coolant concentration, W_w, at the wall surface. It may be observed that especially the level of the free-stream temperature has a strong influence on this relation and that its character is quite different for hydrogen injection than for the constant property situation.

A recovery factor r has been defined by equation (4). The ratio of this recovery factor with hydrogen injection to the recovery factor, r_0, on a solid wall is plotted in Fig. 7 as a function of the coolant concentration, W_w, at the wall surface. It is evident that the level of the free-stream temperature has a strong influence on the recovery parameter, r/r_0. For a free-stream temperature of 123°R, hydrogen injection increases this parameter above a value one.

6. ENGINEERING CORRELATIONS

Gross *et al.* [15] systematically compiled all information which was available at that time for a number of coolant gases and derived empirical relations for skin friction and heat transfer in a binary boundary layer on a "flat plate". With some minor modifications, these equations are:

$$\frac{c_f}{c_{f0}} = 1 - 2{\cdot}08\left(\frac{M_2}{M_1}\right)^{1/3}\frac{\rho_w v_w}{\rho^{*\prime} u_\infty}\sqrt{Re_x^{*\prime}} \qquad (9)$$

$$\frac{q}{q_0} = 1 - 1{\cdot}82\left(\frac{M_2}{M_1}\right)^{1/3}\frac{\rho_w v_w}{\rho^{*\prime} u_\infty}\sqrt{Re_x^{*\prime}}. \qquad (10)$$

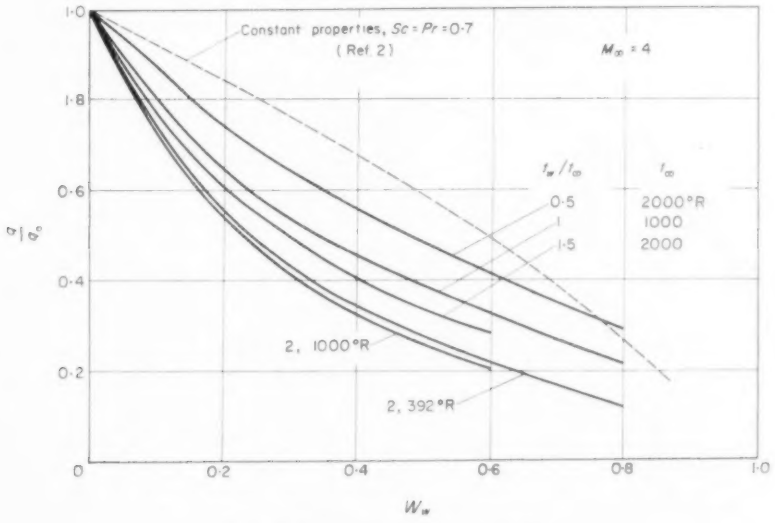

FIG. 4. Hydrogen–air, T_w = const.

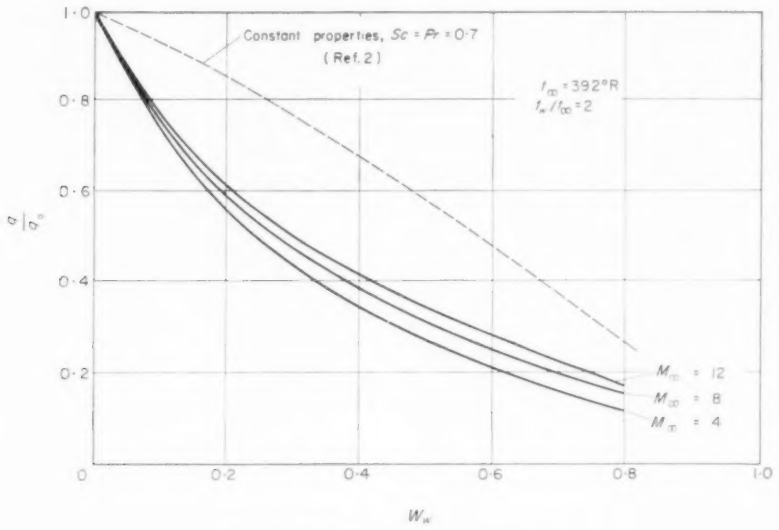

FIG. 5. Hydrogen–air, T_w = const.

The star in these equations indicates that the corresponding property has to be introduced at a reference temperature

$$t^{*\prime} = t_\infty + 0{\cdot}5\,(t_w - t_\infty) + 0{\cdot}22\,(t_{r0} - t_\infty) \quad (11)$$

where the term t_{r0} indicates the recovery temperature of a solid surface without mass injection but with, otherwise, identical boundary conditions. All properties in equations (9) and (10) are those for air. M_1 is the molecular weight of the coolant gas and M_2 the molecular weight of air.

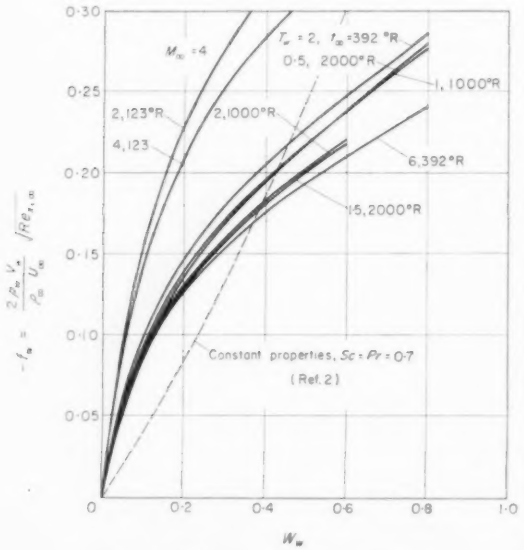

FIG. 6. Hydrogen–air, T_w = const.

Only results of calculations with a free-stream temperature of 392 °R were available at the time when the report [15] was prepared. The analysis in the present paper extended this parameter over a wide range of temperatures. It was, therefore, used to check the accuracy of the above correlations. Figs. 8 and 9 contain the results of our analysis as points, the symbols of which are explained in Table 1. Equations (9) and (10) are indicated as dashed lines, and dashed-dotted lines represent two other correlations which have been developed in [15] specifically for hydrogen as a coolant. The two figures indicate that the general correlations (9) and (10) somewhat underestimate the reduction in friction factor as well as in heat flux, whereas the specific correlations for hydrogen average the results of the present analysis quite well. The deviations from this relation are somewhat larger for the heat flux than for the skin friction. The relation between the mass release parameter, f_w, and the mass

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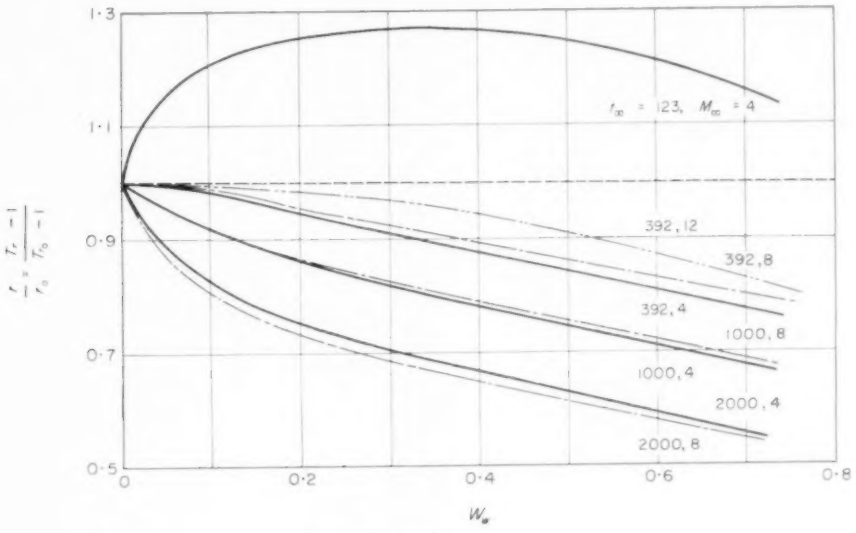

FIG. 7. Hydrogen–air, $T_w = 0$.

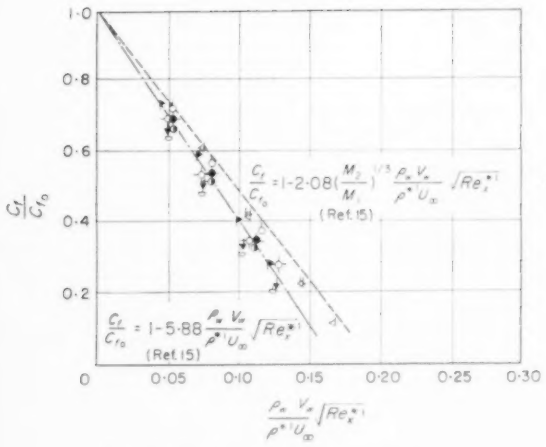

FIG. 8. Hydrogen–air, $T_w = \text{const.}$

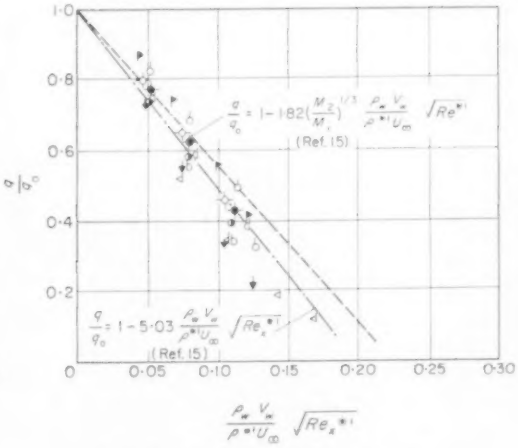

FIG. 9. Hydrogen–air, $T_w = \text{const.}$

fraction, W_w, of the coolant at the wall surface is presented in Fig. 10. A correlation proposed in [15] is indicated in the figure as a solid line. Another solid line connects the analytical results for a free-stream temperature of 123°R. The correlation proposed in [15] gives a reasonable approximation to the analytical results when the values for the free-stream temperature 123°R are excluded.

In summary, it can be stated that equations (9) and (10) approximate the real conditions with an accuracy which is probably sufficient for many engineering calculations. The actual reduction in friction factor and in heat flow with hydrogen as coolant is in an average by approximately 20 per cent larger than the one predicted through these equations. A single correlation as indicated in Fig. 10 can also serve as a reasonable

Table 1. Cooled wall

M_∞	t_∞, °R	$T_w = \frac{t_w}{t_\infty}$	Symbol
0	123	4	◁
0	392	2	◊
0	392	6	♦
0	2000	½	✣
4	123	4	▷
4	392	2	○
4	1000	1	✧
4	1000	2	♦
4	1000	3	○
4	2000	½	▶
8	392	2	□
8	1000	1	□
8	1000	2	◆
8	1000	3	■
8	2000	½	◀
12	392	2	♦ ρ
12	392	6	♦

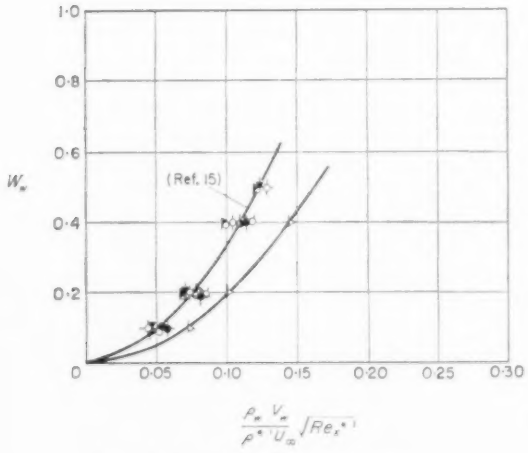

FIG. 10. Hydrogen–air, T_w = const.

approximation to the relation between mass release parameter and coolant mass fraction at the wall when free-stream temperatures below approximately 300°R are excluded.

ACKNOWLEDGEMENT

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REFERENCES

1. E. R. G. ECKERT, Heat transfer and temperature profiles in laminar boundary layers on a sweat-cooled wall, *Rep.* 5646 *HQ Air Material Command, Wright Field, Ohio* (1947).
2. J. P. HARTNETT and E. R. G. ECKERT, Mass transfer cooling in a laminar boundary layer with constant fluid properties. *Trans. Amer. Soc. Mech. Engrs*, **79**, 247 (1957).
3. J. W. SMITH, Effect of diffusion fields on the laminar boundary layer, *J. Aero. Sci.* **21**, No. 3, 154–162 (1954).
4. J. R. BARON, The binary layer associated with mass transfer cooling at high speed, *Naval Supersonic Laboratory, Rep.* 160, *Massachusetts Institute of Technology* (1956).
5. E. A. SZIKLAS and C. M. BANAS, Mass-transfer cooling in compressible laminar flow, in D. J. MASSON (comp.), Mass-transfer cooling for hypersonic flight, *The Rand Corporation, Paper S-51*, June 24 (1957).
6. E. R. G. ECKERT and P. J. SCHNEIDER, Effect of diffusion in an isothermal boundary layer, *J. Aero. Sci.* **23**, 384–387 (1956).
7. E. R. G. ECKERT, P. J. SCHNEIDER, A. A. HAYDAY and R. M. LARSON, Mass-transfer cooling of a laminar boundary layer by injection of a light-weight foreign gas, *Jet Propulsion*, **28**, 34, 39 (1958).
8. A. A. HAYDAY, Mass transfer cooling in a steady laminar boundary layer near the stagnation point, *1959 Heat Transfer and Fluid Mechanics Institute*, University of California, Los Angeles, California, Stanford University Press, Stanford, California.
9. L. LEES, Convective heat transfer with mass addition and chemical reactions, *Combustion and Propulsion, Third AGARD Colloquium, Palermo, Sicily*, March 17–21 (1958).
10. J. P. HIRSCHFELDER, C. F. CURTIS and R. B. BIRD, *Molecular Theory of Gases and Liquids*, John Wiley, New York (1954).
11. W. O. CARLSON and P. J. SCHNEIDER, Transport properties for binary gas mixtures, *University of Minnesota, A.F. Tech. Rep.* 7.
12. J. H. KEENAN and J. KAYE, *Gas Tables*, John Wiley, New York (1948).
13. J. HILSENRATH, *Tables of Thermal Properties of Gases*, National Bureau of Standards—Circular 564, U.S. Government Printing Office, Washington, D.C. (1955).
14. C. F. HANSEN, Approximations for the thermodynamic and transport properties of high-temperature air, *N.A.C.A. Tech. Note* 4150 (1958).
15. J. F. GROSS, J. P. HARTNETT, D. J. MASSON and

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CARL GAZLEY, JR., A review of binary boundary layer characteristics, *Project RAND, Research Memorandum RM*-2516, June (1959).
16. E. R. G. ECKERT, Survey of boundary layer heat transfer at high velocities and high temperatures, *WADC Tech. Rep.* 59-62*h*, Wright Air Development Center, Ohio (1960).
17. E. R. VAN DRIEST, Investigation of laminar boundary layer in compressible fluids using the crocco method, *N.A.C.A. Tech. Note* 2597 (1952).

Résumé—Cet article présente une étude de couche limite laminaire à deux constituants sur une surface avec gradient de pression nul, et avec perte de masse. Le système d'équations aux dérivées partielles donnant la vitesse, la température et le champ de concentration est transformé par similitude en un système d'équations intégrales. On a obtenu des solutions numériques à l'aide d'une calculatrice électronique, pour un écoulement d'air avec injection d'hydrogène pour des nombres de Mach de 0 à 12, des températures de l'écoulement libre de 123 à 2000°R et pour des rapports de température paroi/écoulement libre de ½ à 6. Les solutions dans le cas d'un gradient de température nul dans la couche limite et normal à la paroi, conduisent aux facteurs thermiques pariétaux. Les résultats sont utilisés pour étudier la validité et la précision des relations de caractère technique présentées en [15]. Les corrélations proposées décrivant assez bien la réduction du flux vers la paroi, et du frottement. Aucune corrélation simple n'a été trouvée pour l'influence de la fraction de masse d'hydrogène à la surface sur le taux de perte de masse, et pour le facteur thermique pariétal. Les valeurs obtenues pour une température d'écoulement libre de 123°R diffèrent beaucoup des autres.

Zusammenfassung—Die laminare zwei Komponten-Grenzschicht an einer Oberfläche ohne Druckgradienten und mit Stoffzugabe wird analysiert. Das System der partiellen Differentialgleichungen, die das Geschwindigkeits-, Temperatur- und Konzentrationsfeld beschreiben, ist durch Ähnlichkeitstransformation in ein System von Integralgleichungen übergeführt. Elektronische Digitalrechenmaschinen lieferten die numerischen Lösungen für eine Luftströmung mit Wasserstoffzugabe bei Machzahlen von 0 bis 12, Freistromtemperaturen von 68 bis 1100°K und Temperaturverhältnissen von Wand- zu Freistromtemperatur von ½ bis 6. Die Lösungen für die Bedingung eines Temperaturgradienten Null in der wandnahen Grenzschicht und senkrecht dazu führen auf Temperatur-Rückgewinnfaktoren. Mit Hilfe der Ergebnisse wird geprüft wie weit die in [15] gegebenen Gebrauchsgleichungen gültig und wie genau wie sind. Die dort vorgeschlagenen Beziehungen beschreiben die Abnahme des Wärmeflusses in die Wandoberfläche und die Oberflächenreibung verhältnismässig gut. Für die Abhängigkeit zwischen Wasserstoffanteil an der Wand und Stoffzugabe und für den Temperatur-Rückgewinnfaktor konnten keine einfachen Beziehungen gefunden werden. Die Werte für die Freistromtemperatur 68°K zeigen durchwegs die grössten Abweichungen von den anderen Ergebnissen.

Аннотация—Рассматривается двухкомпонентный ламинарный пограничный слой на поверхности при отсутствии градиента давления и при подаче массы. Система дифференциальных уравнений в частных производных, описывающих поля скорости, температуры и концентрации, сведена к системе интегральных уравнений. На электронно-вычислительных машинах получены численные решения для потока воздуха с вдуванием водорода для чисел Маха от 0 до 12, температур свободного потока от 123 до 2000°R и отношений температур свободного потока и стенки от ½ до 6. Решения при усдовии равенства нулю поперечного градиента температуры на стенке дают возможность вычислить коэффициенты восстановления температуры. Результаты используются для проверки справедливости и точности инженерных формул, приведенных в работе [15]. Предложенные формулы позволяют вычислить уменьшение теплового потока в стенку и, следовательно, определить уменьшение поверхностного трения. Но ещё не удалось получить простой зависимости концентрации водорода на поверхности от скорости вдувания, а также коэффициента восстановления температуры. Для температуры свободного потока 123°R имеют место самые большие отклонения от всех остальных данных.

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BOILING HEAT TRANSFER

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Аннотация—Доклад посвящён проблеме теплообмена при пузырьковом кипении.

Сложность гидродинамических процессов в газожидкостных системах при пузырьковом кипении не дали до сих пор возможности построить полную теорию этого процесса.

В данной работе, исходя из гипотезы о гидродинамической природе кризисов в механизме кипения, сформулирована система дифференциальных уравнений, описывающих процесс пузырькового кипения, и составлена рациональная система характеризующих его критериев.

На основе анализа некоторых принципиальных вопросов теории теплообмена при кипении и составленной системы гидродинамических и тепловых критериев, характеризующих этот процесс, произведено обобщение экспериментальных работ в области пузырькового кипения.

NOMENCLATURE

c,	specific heat capacity;
$D = 2R$,	diameter;
g,	acceleration of gravity;
Δ_i,	difference between mean enthalpy of the flow and enthalpy of the saturated liquid;
L,	length;
p,	pressure;
q,	density of heat flow;
q_{cr},	critical density of heat flow;
r,	latent heat of vapour formation;
T,	temperature;
T'',	saturation temperature above a plane;
w_0,	circulation rate;
x,	gravimetric vapour content of a flow;
α,	heat transfer coefficient;
β,	volumetric vapour content of a flow;
γ,	specific weight;
δ,	transversal linear dimension of a channel;
ϑ,	underheating of the flow nucleus up to the saturation temperature;
λ,	thermal conductivity;
ν,	coefficient of kinematic viscosity;
σ,	coefficient of surface tension;
τ,	time.

Indices

'',	vapour phase;
int,	interface.

1. INTRODUCTION

BOILING heat transfer plays an essential part in solving a number of new problems and is widely used in the power industry. However in these cases the levels of formation of heat transfer surfaces can differ by some orders of magnitude. Thus, for example, when freon boils the heat fluxes are 10^3 kcal/m²/h in refrigeration technique, 10^4 in evaporators of electric power stations, 10^6 in water-moderated water-cooled reactors and 10^7 kcal/m²/h in elements of jet engines.

In connexion with this the value of critical density of heat flux is of decisive importance for reactors and other highly forced installations: for this value determines the highest limit of safe working of an assembly.

In industrial power installations the absolute values of heat transfer coefficients are of great importance, since they are usually of the same order of magnitude both for heating and heated media.

Film boiling takes place in hardening processes (heat treatment), at near critical pressures, and in some other specific cases.

The present paper deals with the problems of heat transfer with nucleate boiling.

Nucleate boiling is characterized by the origin of vapour in various places on the heating surface called nuclei of vapour formation. It creates a rather complex heterogeneous structure of liquid and vapour phase. This phenomenon relates to one of the most complicated problems of the hydrodynamics of gas–liquid systems, in particular to flows in which phase components of the flow are split up in separate units bounded by moving interface surfaces. The number of these units (bubbles, drops, films) variable in space and time is great, and therefore probability laws of multicomponent systems must be valid here.

At present there are no consistent analytical methods for such systems.

The purpose of the present work is to discuss some fundamental problems of the theory of heat transfer with boiling, and to generalize the results of experimental work.

2. ORIGIN OF VAPOUR BUBBLES

The intensity of heat transfer with nucleate boiling is conditioned by the interaction of convection, connected with the directed motion of the whole flow of a liquid, and local convection, caused by the origin, growth and rise of vapour bubbles. The latter process is of decisive importance at sufficiently high flux densities, i.e. at considerable rates of vapour formation.

The probability of the creation of a vapour phase in a superheated liquid is proportional to the number e raised to the power $1/-(\Delta t)^2$. Consequently vapour bubbles should rise just near the heating surface where the superheating of the liquid is the highest in comparison with the saturation temperature above the surface.

In some works on heat transfer with boiling an erroneous viewpoint of Kruzhilin [21, 22] has prevailed, namely that vapour bubbles rise on the projections of roughness. At present it may be considered to be established that the centres of boiling on a solid surface are hollows, pores and concavities near the base of protruberances and roughness. Apparently for the first time this mechanism was pointed out by Andreev [5], who studied cylindrical pores. This mechanism was then studied in the works of Nesis [33], Courty and Foust [57]. The rise of a vapour bubble on a conical pore was studied by Courty and Foust [57], Bankoff [54] and others. A review of these works was made by Zuber [72].

Thus the process of boiling is conditioned not only by the probability of bubble rise at a given degree of superheating of the liquid, but also by the probability distribution of centres of vapour formation on the heating surface. The probability character of the rise of vapour bubbles on a solid surface is proved distinctly by the curves of bubble distribution in accordance with the frequencies of their formation, which were obtained at certain pressures by Zysina and Kutateladze [15]. Similar curves were later obtained by Treshchev [48] for boiling by heating the nucleus of a liquid flow to the saturation temperature. The diameter of bubbles just leaving the heating surface is also not a constant value, but obeys some distribution law. Under static conditions the diameter of a bubble just leaving a smooth heating surface is determined according to Fritz [60] by equation:

$$D_0 = 0{\cdot}02\theta \sqrt{\left(\frac{\sigma}{\gamma - \gamma''}\right)}, \qquad (1)$$

see [23].

The variability of D_0 points to the fact that the value of the marginal angle θ^0 cannot be considered as a distinct characteristic of a large area of a heating surface. Moreover recent cone-filming carried out at the Institute of Thermal Physics showed that when the boiling is intense a bubble has a complex form before breaking away, and that often it is bound to the heating surface by a cylindrical neck.

The facts cited above show distinctly that the hydrodynamic and heat processes going on with bubble boiling can only be statistically represented.

3. VAPOUR BUBBLE GROWTH RATE

The growth of a vapour bubble is connected with the density of the heat flow on a bubble surface by the equation

$$\frac{dR}{d\tau} = \frac{q_{int}}{r\gamma''}. \qquad (2)$$

Bošnjaković [55] applied this relation for determining the value q_{int} in accordance with the

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experimental data of Jacob and Linkew [64] and Jacob [65].

If a liquid medium round a bubble is regarded as stationary, then

$$q_{\text{int}} = C(T - T_{\text{int}})\sqrt{\left(\frac{\lambda c\gamma}{\tau}\right)}, \qquad (3)$$

where the proportionality factor C is equal to $\sqrt{3}$ according to Plesset and Zwick [66] and to $-\pi/2$, i.e. to $C \approx 1{\cdot}6$ according to Forster and Zuber [59]; T is the temperature of a superheated liquid, and T''_{int} is the saturation temperature at the boundary of a bubble.

According to this formula, for the conditions of formation of one of the vapour bubbles photographed by the author and Zysina during the experiments, we have:

τ_{sec}	0·00475	0·0095	0·0285
q_{int} kcal/m²/h	203 000	143 000	83 000

Under the conditions of thermodynamic equilibrium for a spherical bubble

$$T''_{\text{int}} = T'' + \frac{2\sigma\gamma}{R(\gamma - \gamma'')} \cdot \frac{dT''}{dP}. \qquad (4)$$

While heat transfer is going on there is no thermodynamic equilibrium, a bubble grows due to liquid evaporation into it, and the value of T_{int} differs from that in equation (4). Assuming that there is only vapour in a bubble we may, in the first approximation, connect the rate of evaporation with the pressure drop in the vapour boundary layer by the known formula of the kinetic theory:

$$q_{\text{int}} = \xi r \Delta P \sqrt{\left(\frac{gm}{2\pi BT''}\right)}, \qquad (5)$$

where $\xi \leqslant 1$ is the accommodation coefficient, B is the universal gas constant, m is the molecular weight.

The corresponding change in saturation temperature is

$$\Delta T''_q = \frac{q_{\text{int}}}{\xi r}\sqrt{\left(\frac{2\pi BT''}{gm}\right)} \cdot \frac{dT''}{dP}. \qquad (6)$$

Summing up equation (4) and (6) we find that

$$T''_{\text{int}} = T + \left[\frac{2\sigma\gamma}{R(\gamma - \gamma'')} + \frac{q_{\text{int}}}{\xi r}\sqrt{\left(\frac{2\pi BT''}{gm}\right)}\right]\frac{dT''}{dP}, \qquad (7)$$

where

$$\frac{dT}{dP} = \frac{AT''(\gamma - \gamma'')}{r\gamma\gamma''}.$$

In technical units $A = \frac{1}{427}$.

The comparison of terms in square brackets of equation (7) is given in Fig. 1. It is distinctly seen from this figure that even at $\xi = 1$ the correction for the evaporation kinetics is not only commensurable with the correction for the curvature of the interface but in a number of cases it may exceed the latter.

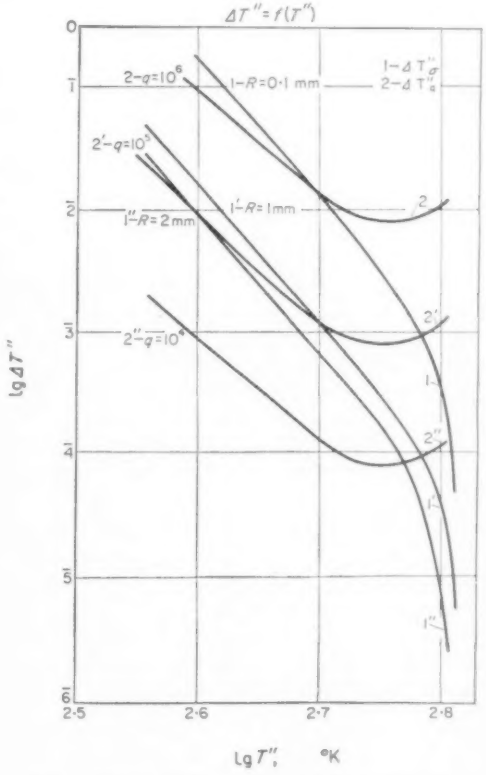

FIG. 1. Values of $\Delta T''_\sigma$ and $\Delta T''_q$ by equation (7).

4. FIRST CRITICAL HEAT FLUX DENSITY IN PIPES AND CHANNELS

It is usual to begin the study of boiling processes with the problem of free convection in a large volume of a liquid. From the technical standpoint, however, the most essential thing is the forced flow of a boiling liquid in tubes and channels.

At present vast experimental material has accumulated on critical heat flows in tubes and to a lesser degree, on critical heat flows in slotted channels.

Elucidation of the influence of pulsations on the value q_{cr} was of great importance. The number of regimes and thermal parameters determining the maximum heat flux density at which the nucleate boiling and, accordingly, a high intensity of heat transfer are still maintained is very high. In connexion with this the method of a generalized analysis of the phenomenon in question and the rational forms of connexion between the values determining it are of paramount importance.

Two viewpoints concerning the critical boiling in a large volume of a liquid were put forward.

According to the first viewpoint proposed by the author [24, 25] the transition from nucleate to film boiling is considered to be a qualitatively specific process, the basis of which involves the loss of the stability of structure of a two-phase boundary layer which had existed before the onset of the crisis.

Every critical hydrodynamic condition, as in this case the transition from one regime of boiling to the other, is determined not by a strictly fixed quantity of a characteristic value (q_{cr} in the present case) but only by a most probable value.

The second viewpoint most lucidly formulated by Kruzhilin [21], is based on the assumption that no new qualitative pecularities whatever arise during the first crisis in the boiling regime and, consequently, the critical heat flux is determined by the same initial sum of the generalized parameters (similarity criteria) by which the coefficient of heat transfer with bubble boiling is determined. This hypothesis was not developed for a forced flow but when applied to boiling with free convection, as shown in [28], it led to a very complicated empirical formula.

According to the hydrodynamic hypothesis the transition from nucleate to film boiling begins with loss of structural stability of a two-phase boundary layer in one of the small regions of the heating surface and from the formation of a steady vapour "spot". The rise of temperature under this "spot" caused by the drop of the heat transfer coefficient leads to a further development and distribution of film boiling over the heating surface. Such a concept fully corresponds to the phenomenon observed with boiling on heat resisting (e.g. graphite) rods and plates.

There is no basic difficulty in writing the conditions of mechanical interaction between a vapour and liquid for a differential element of the interface of such a system. And as far as the local stability of the structure is concerned the question of boundary conditions for the whole system does not arise. Accordingly, the system of equations consists of the equations of motion and continuity for each of the phases and for conditions of mechanical interaction of phases on the interface [25].

Considering, in the limit, a non-viscous liquid it is also possible to use the mechanism of the stability theory of Rayleigh's two-layer flow [39].

With such an interpretation of the problem it is the critical velocity of the vapour which has to be determined, and the question arises of the relationship between the local quantity of this value and the average heat flux on the heating surface.

It is evident that the only real measure of vapour formation rate is the value

$$\frac{q}{r\gamma''} \tag{8}$$

representing the maximum vapour volume which can be formed per unit time on unit heating surface.

We consider here the maximum vapour volume since in a general case a fraction of the heat flow q may not be spent on vapour formation (e.g. in the flow where the nucleus is underheated to the saturation temperature).

From the heat balance of the mixture one

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more characteristic appears which can be presented in the form of the criterion:

$$K = \frac{r}{\Delta i} \tag{9}$$

where Δi is the difference between the mean flow enthalpy and the enthalpy of the saturated liquid.

In the flow of saturated liquid $1/K = x$ where x is the vapour content by weight, i.e. in this case the criterion K actually has a purely hydrodynamic meaning.

When $\bar{T} < T''$, then the criterion $K = r/c\vartheta$ where $\vartheta = T'' - \bar{T}$ is the underheating of the flow nucleus to the saturation temperature. In this case K has a purely hydrodynamic meaning only with fully turbulent heat transfer and serves as a measure of the vapour flow decrease due to the condensation process in a cool nucleus. If the molecular heat conduction is important, then K as the heat criterion is connected with the Prandtl number. This question will not be considered here.

The stability of the two-phase system is always affected by the dynamic head of the phase which is being produced, the scale of which in the given case is the value

$$\left(\frac{q}{r\gamma''}\right)^2 \frac{\gamma''}{q}, \tag{10}$$

as well as gravity and surface tension. As the scale of the joint action of the two latter forces [taking into account equation (1)] the following value can be taken

$$(\gamma - \gamma'') \sqrt{\left(\frac{\sigma}{\gamma - \gamma''}\right)}. \tag{11}$$

The square root of the ratio of these values gives the criterion [24, 25]

$$K = \frac{q}{r\sqrt{(q\gamma'')}\sqrt[4]{[\sigma(\gamma - \gamma'')]}}. \tag{12}$$

At $q = q_{cr}$ the criterion of equation (12) can be determined.

If we use the Rayleigh stability criterion then the criterion K will be of the form

$$K = \frac{q_{cr}}{r\sqrt{\left[q\gamma''\left(1 + \frac{\gamma''}{\gamma}\right)\right]}\sqrt[4]{[\sigma(\gamma - \gamma'')]}}. \tag{13}$$

The value of this expression is marked with the index 1 for the first critical density of the heat flow (the rise of film boiling) and with the index 2 for the second critical density of the heat flow (the destruction of the vapour film).

The difference between equations (12) and (13) by the value $\sqrt{[1 + (\gamma''/\gamma)]}$ is not so essential, and is connected with the solution of the initial system of equations adopted by Rayleigh.

It was shown in the works of the author [24, 25, 26] that in the range of the assumptions considered, the following relation exists between the dimensionless criteria:

$$K = \Phi\left[w\left(\frac{\gamma - \gamma''}{g^2\sigma}\right)^{1/4};\ \frac{v^2}{g}\left(\frac{\gamma - \gamma''}{\sigma}\right)^{3/2};\right.$$
$$\left.\frac{\gamma}{\gamma''};\ \delta\left(\frac{\gamma - \gamma''}{\sigma}\right)^{1/2}\frac{L}{\delta};\ \frac{\Delta i}{r}\right]. \tag{14}$$

Moreover, the law of heat supply distribution along the heating surface should be predicted.

For the underheated liquid the hydrodynamic hypothesis led to the conclusion that there is a linear relation (at any rate in the first approximation) between the criteria K and $\Delta i/r$ of the form:

$$\frac{q_{cr,\,\vartheta}}{q_{cr,\,0}} = 1 - B\left(\frac{\gamma}{\gamma''}\right)^n \frac{\Delta i}{r}. \tag{15}$$

At a purely turbulent exchange the coefficient B is a certain constant. In the case of the influence of molecular heat conduction on heat transfer near the freshly forming elements of a vapour phase [e.g. by equation (3)] the coefficient B is the function of Pr and Ar.

As regards the experimental data we have to note that in spite of their multiplicity the agreement between the results of various investigations is still unsatisfactory. This criticism applies to qualitative results, apart from quantitative discrepancies which can be explained by the inherent instabilities of critical constants.

A number of discrepancies were explained after the role of the initial volumes of vapour bubbles in the rise of pulsations and the influence of the length of the heated section were revealed [4, 11, 46].

Most experiments with slotted channels and

tubes show that the influence of the slot width or diameter is small at:

$$\delta > (1 \div 3) \sqrt{\left(\frac{\sigma}{\gamma - \gamma''}\right)}. \qquad (16)$$

The linear relation between q_{cr} and Δi in the region of underheating $\vartheta > 0$ is satisfactorily proved by the experimental data for sufficiently free channels at pressures up to the critical ones. Povarnin and Semenov [36] confirmed that the relation was satisfactory for liquid flow rates up to 40 m/s.

The experiments carried out by Zenkevich, Subbotin, Styrikovich, Miropolsky, Chirkin *et al.* [12, 13, 17, 46, 52] proved distinctly the general character of equation (15). However, the numerical values of the coefficients B and n are not equal in different experiments.

The experiments of Zenkevich [13], Miropolsky and Shitsman [17] give a clear dependence of q_{cr} on w_0 at $\vartheta \geqslant 0$ which is satisfactorily described by the empirical formula [24, 25]:

$$K_{10} = 0{\cdot}023\, w_0^{0{\cdot}5} \left(\frac{\gamma - \gamma''}{g^2\sigma}\right)^{0{\cdot}125} \qquad (17)$$

where K_{10} is the value of the complex K at $\vartheta = 0$.

The experiments carried out by Chirkin and Yukin, Averin, Kruzhilin and Kutateladze with water flow in broad annular slots are also satisfactorily described by an equation of the type (17) but the proportionality factor is equal to 0·085 [25, 26]. The experiments of Ornatsky [35] are quantitatively 30 per cent lower.

The experiments of Doroshchuk and Frid [11, 17] show a weaker influence of the velocity of a liquid flow on the value of q_{cr}, and some influence of w_0 on the coefficient B is revealed. The experiments of Aladiev and Dodonov [3] differ in this respect from the results of other investigators. In these experiments the influence of w_0 on q_{cr} is practically zero at low pressures, while at high pressures it is comparatively weak and the monotonous relation between the coefficient $B(\gamma/\gamma'')^n$ and pressure is disturbed.

Some discussion on the relation between q_{cr} and w_0 at high velocities of a liquid flow is given in the work by Povarnin and Semenov [36].

In the range $x > 0$, i.e. in the flow of a saturated vapour–liquid mixture, the dependences for q_{cr} are still more complicated. It is positively established that the degree of influence of vapour content depends on the gravimetric mixture velocity or, in other words, on the so-called "circulation rate"

$$w_0 = \frac{G_{mix}}{\gamma\Omega}, \qquad (18)$$

where G_{mix} is the gravimetric mixture rate per second, Ω is the cross-section area of the channel.

When the pulsations are absent

$$\left(\frac{\partial q_{cr}}{\partial x}\right)_{w_0} < 0$$

i.e. the critical density of the heat flux decreases despite the essential rise in the average liquid flow rate at the increase in the vapour content of the mixture. Comparing this result with equation (17) it is necessary to conclude that the increase in vapour content of the flow essentially changes the vapour content of the layer near the wall, and favours the destruction of liquid structures existing in the layer.

On the basis of the experimental data available, more or less distinct relations may be outlined only for the region of volumetric vapour contents $\beta < 0{\cdot}9$. Since the volumetric and gravimetric vapour contents are connected by the formula

$$\frac{\beta}{1-\beta} \cdot \frac{\gamma''}{\gamma} = \frac{x}{1-x} \qquad (19)$$

then at low pressures very small weight vapour contents correspond to the values of $\beta < 0{\cdot}9$ while at high pressures they are of the order of 0·1 and more.

Rybin treated the experimental data obtained by Aladiev, Buchberg, Doroshchuk, Miropolsky, Subbotin and others on tubes of 9 mm more in diameter, vapour contents of $0 < \beta$ or $< 0{\cdot}9$ and entrance conditions under which the water entering the experimental section was slightly underheated to the saturation temperature. The results for the two values of the criterion:

$$F_* = w_0 \left(\frac{\gamma - \gamma''}{g^2\sigma}\right)^{1/4} \qquad (20)$$

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are given in Fig. 2 from which both the absolute discrepancies in the data of some investigators and the general scattering of experimental points are distinctly seen.

It is interesting to note that the average lines at the point $\beta = 0$ practically coincided with the results of calculations by equation (17).

In Fig. 3 the averaging straight lines are compared for various values of the criterion of equation (20). As is seen, the relative influence of vapour content weakens with the decrease of this criterion.

The experimental material available for the region of high weight vapour contents ($\beta > 0{\cdot}9$) is insufficient for generalizations.

Thus systematization and further accumulation of experimental data on flow in tubes and channels is essential before reliable calculation formulae for the value q_{cr} can be obtained. In connexion with this the complicated empirical

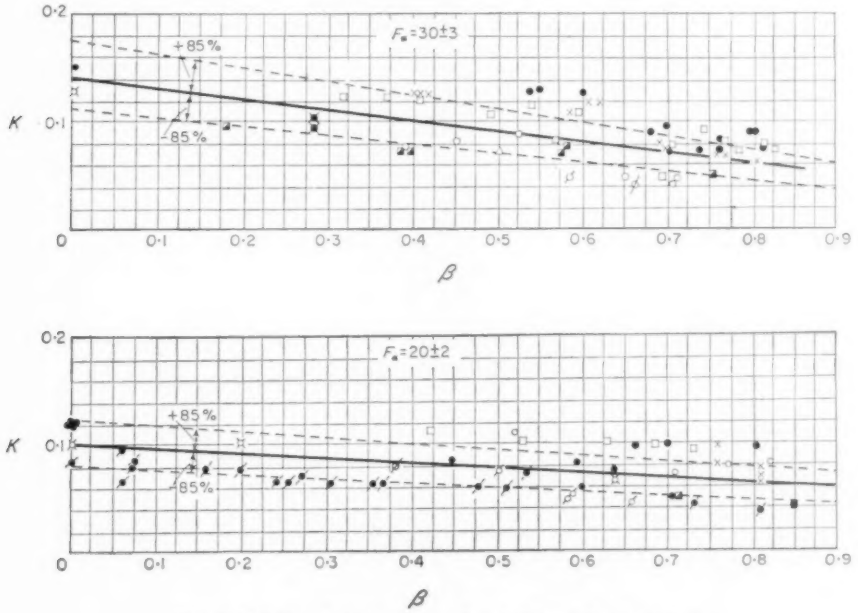

FIG. 2. Relation between K_1 on β and F_* in a flow in tubes.

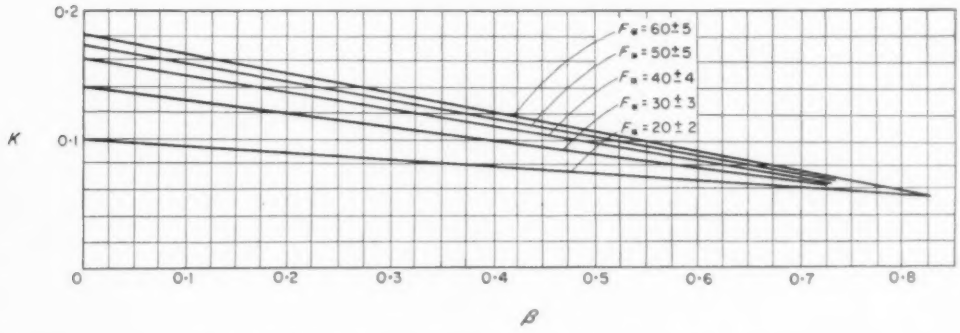

FIG. 3. Relation between K_1 on β and F_* $\beta < 0{\cdot}9$ (mean lines).

equations proposed by Zenkevich [13] and Ivashkevich [16] may be considered only as the first and not very successful experiment.

5. CRITICAL DENSITY OF HEAT FLOW AT FREE CONVECTION IN A LARGE VOLUME OF LIQUID

The results of the application of the hydrodynamic theory of control boiling to the conditions of free convection in a large volume of liquid are well known and reduce to the following expressions [24, 25]:

$$K_{10} \approx \text{const.}; \tag{21}$$

$$\frac{q_{cr,2}}{q_{cr,1}} < 1. \tag{22}$$

The influence of underheating of the bulk of a liquid is expressed by equation (15).

Using the Rayleigh stability theory of two-layer flow Zuber and Tribus [71] found that in non-viscous liquid

$$\left.\begin{aligned} &0{\cdot}12 < K_{10} < 0{\cdot}16 \\ &B = 5{\cdot}3\,(Pr^2\,Ar_*)^{-1/4} \\ &n = \tfrac{3}{4} \end{aligned}\right\} \tag{23}$$

Equation (3) was also used for calculating the latter two coefficients.

It is interesting to note that the group B in equation (23) changes only very slightly. The theoretical values (23) coincide closely with the majority of the experimental data. However it is important to pay attention not to this good qualitative and quantitative corroboration of the theory by the majority of experimental investigations but to some discrepancies revealed. Apparently, if these deviations are not connected with experimental inaccuracy and inaccurate data on physical properties of the material investigated, they must serve as the starting point for further development of the theory. Here we mean small (less than 0·1) values of K_1 for melted salts revealed in experiments of Robin

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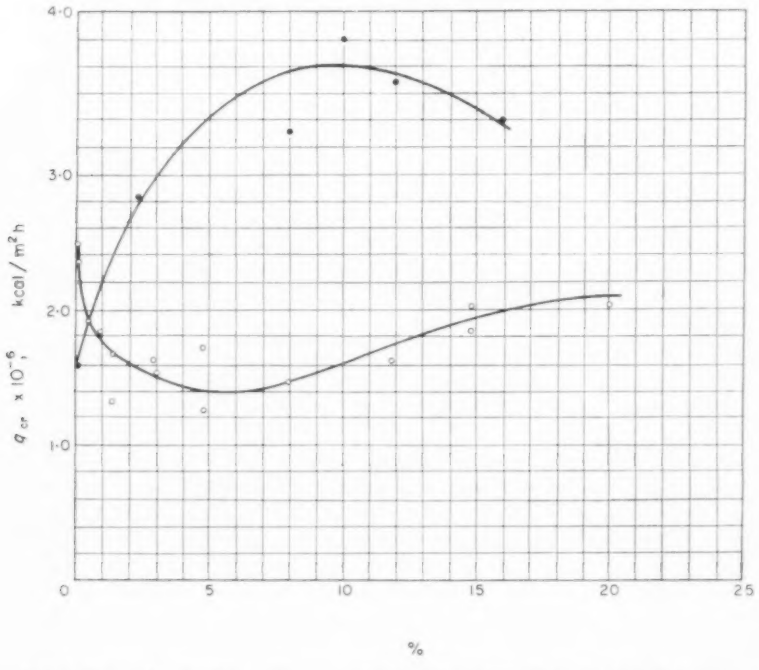

FIG. 4. Dependence of $q_{cr,1}$ on concentration of mixture of butyl alcohol in water:
●—wire, $p = 9{\cdot}8$ atm ○—plate, $p = 11$ atm.

[38] (with the approximate conservation of the values K_2) and the deviations of values B and n in the experiments for diphenyl from those for water and alcohol.

Highly interesting results in critical heat flows in mixtures with boiling on thin wires were obtained by Van Wijk, Van Stralen [69], Fastovsky and Artym [51]. It was found that the value of $q_{cr,1}$ essentially exceeded the corresponding values for each of the components at a certain composition of a binary mixture. For pure components the values of the critical heat flux were obtained coinciding well with equation (21) when a correction for viscosity was introduced according to Borishansky [6]

$$K_{10} = 0{\cdot}13 + 4\,Ar_*^{-0{\cdot}4}. \tag{24}$$

The mechanism of this phenomenon is still far from clear. Moreover the experiments carried out by Bobrovich *et al.* in the Institute of Thermal Physics with heating surfaces greater than those used in the investigations mentioned did not discover this effect in any particular form (Fig. 4).

6. INFLUENCE OF FORCED CONVECTION ON HEAT TRANSFER WITH NUCLEATE BOILING IN TUBES

The influence of the directed convection characterized by the "density-displacement" velocity and vapour content of the flow on the intensity of heat transfer with nucleate boiling depends both on the density of the heat flow and the physical properties of the heat agent. Great attention is devoted to this problem in the works by Sterman, Styushin, Sorokin and others [18, 41–44].

With nucleate boiling the flow rate has little influence on the heat transfer coefficient at high pressure and powerful heat flow, as confirmed by the experiments of Aladiev, Tarasova *et al.* [2, 47].

At low pressures and moderate heat flows this influence may be quite essential.

As was shown by the author in [24, 29] the influence of circulation velocity on the heat transfer coefficient with bubble boiling can be estimated by the ratio of the heat transfer coefficient α_0 in the flow of a non-boiling liquid to α_{00} with a developed boiling.

For the region where there is no vapour content influence we have

$$\frac{\alpha}{\alpha_0} = f\left(\frac{\alpha_{00}}{\alpha_0}\right), \tag{25}$$

and moreover

$$\left.\begin{aligned} &\frac{\alpha_{00}}{\alpha_0} = 0, \quad f = 1, \quad \dot{f} = 0; \\ &\frac{\alpha_{00}}{\alpha_0} \to \infty, \quad f \to \frac{\alpha_{00}}{\alpha_0}, \quad \dot{f} \to 1. \end{aligned}\right\} \tag{26}$$

The simplest interpolation formula satisfying the conditions of (26) is

$$\frac{\alpha}{\alpha_0} = \sqrt[n]{\left[1 + \left(\frac{\alpha_{00}}{\alpha_0}\right)^n\right]}. \tag{27}$$

The dependence of α on w_0 and q is shown in Fig. 5 according to test runs carried out by Sterman. It is distinctly seen from this figure that the influence of the circulation rate decreases with the increase in q. The treatment of these test runs shows that the value $n = 2$ convenient for practical calculations can be adopted [29].

The value α_{00} representing the heat transfer coefficient in developed boiling is determined for the given liquid by the equation:

$$\alpha_{00} = c(p)q^{0{\cdot}7}. \tag{28}$$

For the experiments with water boiling in tubes the values of the pressure function $c(p)$ is shown in Fig. 6 according to the data by Aladiev, Sterman *et al.*, Tarasova *et al.* [2, 44, 47]. This dependence is qualitatively similar to that obtained by Borishansky [7] in experiments with boiling at free convection in a large volume of liquid but it is displaced somewhat lower. This quantitative discrepancy of some 20–30 per cent, is hardly fundamental in character.

The considered expressions are valid with $\beta < 0{\cdot}7$. It should be noted that the experimental material available is not plentiful for these values of vapour contents. The data available for the region $\beta > 0{\cdot}7$ are insufficient for any generalization to be made.

The experiments of Tarasova *et al.* [47] carried

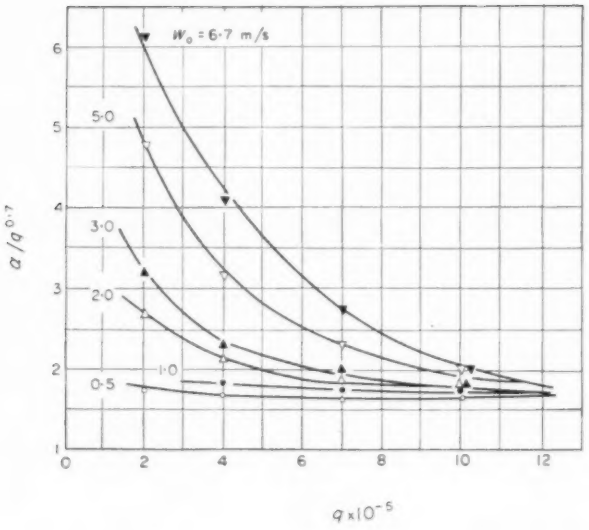

FIG. 5. Relation between $\alpha/q^{0.7}$ and q and w_0 for water, $p = 2$ atm, tube 16 mm in diameter.

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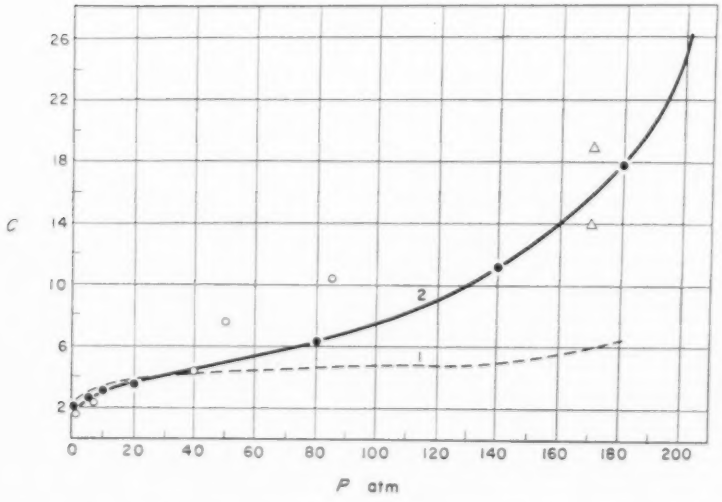

FIG. 6. The value of c_p in formula (28) from experiments with water boiling in tubes:

● according to [2]
○ according to [44]
△ according to [47]

Curve 1 by the Kruzhilin formula.
Curve 2 by the Kutateladze formula. In these two formulas the proportionality factors are diminished by 20 per cent.

out at 170 atm showed that at $\beta > 0{\cdot}7$ some increase at the beginning and then strong decrease of the heat transfer coefficient was observed.

7. HEAT TRANSFER WITH DEVELOPED NUCLEATE BOILING

The absence of any influence of velocity and channel sizes on the heat transfer coefficient with developed nucleate boiling (at least at $\beta < 0{\cdot}7$ and $D > 3\sqrt{[\sigma/\gamma - \gamma'']}$) is connected with the fact that in this case the sweeping of the heating surface by the liquid is fully determined by the process of vapour formation. This is another confirmation of the qualitative difference between the mechanisms of heat transfer with nucleate boiling and the appearance of film boiling, since the value q_{cr} depends quite distinctly on the velocity whereas α hardly depends on it.

Thus with developed boiling the heat flux q alone is the regime factor with respect to the heat transfer coefficient. Moreover there is the influence of the complex of physical properties of the medium varying along the saturation line.

The probability character of the distribution of the vapour formation centres over the heating surface and of the vapour bubble generation process itself creates the instability of the hydrodynamic and heat conditions in the region close to the wall as already mentioned above.

In connexion with this the values of heat transfer coefficients are far from being equal despite various experiments which were adequate in themselves. Due attention is not paid to this circumstance and some investigators still only consider their own values of α to be reliable.

When determining the similarity criteria of the heat transfer process it is necessary to study the interaction of the whole flow with the heating surface, i.e. one should not limit oneself to the investigation of local conditions, as was the case in the problem of the critical regime. This circumstance creates some difficulties, not yet overcome, for formulating the corresponding system of equations and in addition to them, the conditions on the boundary of the system.

However, these difficulties differ slightly from similar ones which occur with the analytic formulation of conventional turbulent flow, which also consists of a statistical sum of finite formations, variable in space and time.

Since at boiling there are parts of the medium with different physical properties (liquid and vapour), the mathematical formulation of the problem must contain the relations both for the phenomena inside each of the regions and for the interactions (mechanical and thermal) on their surfaces, i.e. at the interfaces.

The conditions on the boundaries of the whole system, i.e. on the heating surface should be predicted in an independent manner.

All the interactions taking place in a two-phase flow of any complexity are direct consequence of the total effect of local interactions. Therefore if we assume that this totality does not introduce any specific physical properties, the mechanism for the derivation of the similarity criteria is made up of the system of differential equations of motion and heat conduction in each phase, the conditions of local interactions on the interface and of the conditions on the heating surface (boundary conditions).

Essentially, this problem was raised in the same form by the author [23, 24] and later on by Eigenson [53].

This approach to the derivation of the similarity conditions can be methodically formulated in two ways:

(1) The equations of heat transfer and motion in a differential form for each phase, the equations of the interaction on a differential surface element of the interface and the system boundary conditions are written out;

(2) The equations of heat transfer and motion in the differential form are written out for the phase which is of decisive importance for the process (for the liquid in nucleate boiling); the second phase is considered in the form of finite elements of definite geometry (e.g. spherical vapour bubbles) the equations of motion of the centre of gravity of such an element and heat balance equations are written out, the system boundary conditions being the same as in the first case.

In a like manner the system of equations can be made up when considering a new phase to be a heat and mass sink [10, 42]. Such a procedure does not differ in essence from the one

above but in this case the discrete elements of the new phase should be considered as material points. Therefore such a method may be applied when the region where the process is going on is very large in comparison with the elements of the dispersed phase.

There is no basic difference between the two methods since the motion equation of the bubble centre of gravity can be obtained by solving the equation system of the complete motion of both phases. Such solutions were obtained in the finite form for some ranges of Reynolds numbers by Hadamard [61] Rybchinsky [68] and Levich [30].

Naturally the system of equations should correspond to the adopted model of the real physical process and be closed in the sense that the number of equations corresponds to the number of unknowns entering them. The degree of the model conformity with the real process is in the end given by the comparison between experimental data and the results of generalizations. This circumstance is not of course peculiar to the boiling process by has general significance [19].

The consideration of equations of liquid motion and heat conduction with regard to the equations of motion and mechanical and thermal interaction of a vapour phase gives three criteria:

$$Re_* = \frac{q}{r\gamma'' \nu}\left(\frac{\sigma}{\gamma-\gamma''}\right)^{1/2};$$

$$Ar_* = \frac{g}{\nu^2}\left(\frac{\sigma}{\gamma-\gamma''}\right)^{3/2}\left(1-\frac{\gamma''}{\gamma}\right);$$

$$Pr = \frac{\nu}{Q}. \qquad (29)$$

Here the value $q/r\gamma''$ is the velocity scale and the value $\sqrt{(\sigma/\gamma - \gamma'')}$ is the scale of the linear dimension.

At highly intensive boiling the layer near the wall may be so turbulized that the influence of the molecular viscosity may be neglected in the equation of motion. In such a case we obtain only two criteria as distinct from (29):

$$Pe_* = Pr\, Re_* = \frac{q}{r\gamma'' Q}\left(\frac{\sigma}{\gamma-\gamma''}\right)^{1/2};$$

$$Pr^2\, Ar_* = \frac{g}{Q^2}\left(\frac{\sigma}{\gamma-\gamma''}\right)^{3/2}\left(1-\frac{\gamma''}{\gamma}\right). \qquad (30)$$

In particular, this is valid for the media with $Pr \ll 1$. It follows from boundary conditions that:

$$Nu_* = \frac{a}{\lambda}\left(\frac{\sigma}{\gamma-\gamma''}\right)^{1/2}. \qquad (31)$$

The last criterion was introduced by Jacob [65].

The continuity condition of the temperature field on the interfaces has the form:

$$T_{\text{int}_1} = T_{\text{int}_2} = T''_{\text{int}}. \qquad (32)$$

If T''_{int} is determined by equation (4), as Kruzhilin determined it [21], then a new criterion is obtained which can be reduced to the form:

$$K_t = \frac{(r\gamma'')^2}{Ac\gamma T''\sqrt{[\sigma(\gamma-\gamma'')]}}. \qquad (33)$$

If the kinetics of evaporation is taken into account as was done by Zysina and Kutateladze [14, 24], the following criterion appears:

$$K_p = \frac{p}{\sqrt{[\sigma(\gamma-\gamma'')]}}. \qquad (34)$$

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As was shown above, the terms of the equation which lead to these criteria are at least commensurable.

If we assume that $dp/dT'' \approx p/T''$ then:

$$\frac{K_t}{K_p} \approx K_s\frac{\gamma''}{\gamma}\left(1-\frac{\gamma''}{\gamma}\right) \qquad (35)$$

where:

$$K_s = \frac{r}{cT''} \qquad (36)$$

is the modification of the criterion K introduced by Sterman.

The treatment of experimental data with simultaneous use of the criteria K_t and K_p (or K_p and $K_s\ \gamma''/\gamma$) is difficult. However it turns out that there is nearly a single-valued relation between these criteria for many liquids. And this explains the fact that the empirical formulae built by the limited system of criteria as:

$$Nu_* = f_1(Pr;\quad Re_*;\quad K_p)$$

$$Nu_* = f_2(Pr;\quad Re_*;\quad K_t), \qquad (37)$$

give more or less satisfactory results.

In practice for a particular liquid one treatment is somewhat preferable.

Tobilevich [49] and some other authors showed that the treatment with the criterion K_p gives slightly better results especially in the region of high pressures. In particular this refers to water (Fig. 6).

Some authors striving for better agreement between experimental data developed formulae which contain a limited number of criteria and individual constants for each liquid. These constants are even commensurable in some formulae. In this connexion we may mention the formulae of Rohsenow [67], Aladiev [2], Jacob and Linke [64], Tolubinsky [50] and Levy [62]. Apparently, in this case it is preferable to have simple empirical formulae for the given substance. Aladiev suggested a good form of such a formula for water. It is expedient to give this formula as:

$$a = \frac{c_q^{0.7}}{1 - (2{\cdot}56 \times 10^{-3} t'')}, \qquad (38)$$

where c is the constant characterizing the heating surface. With boiling in tubes $c \approx 2{\cdot}1$; with boiling in a large volume on brass tubes $c \approx 2{\cdot}6$ and $c \approx 3$ with boiling in a large volume on tubes of stainless steel.

In this respect the methods of developing empirical relations on the basis of the concepts on the states of the substance are useful. This subject is considered by Novikov [34], Borishansky [8], Rychkov [40] and others.

CONCLUSIONS

(1) The available theoretical and experimental data on bubble boiling mechanism concern, in the main, the conditions under which bubbles interact slightly with one another. With boiling of a saturated liquid at high thermal flux densities the development mechanism of vapour bubbles changes considerably. Further investigations are needed for creating a sufficiently complete theory of nucleate boiling.

(2) The hypothesis of hydrodynamic nature of crises in boiling mechanism and crisis development from local disturbances of the stability of a two-phase boundary layer allowed a strict formulation of the system of differential equations describing this process, the choice of both a rational form of the criterion determined and the system of principal determining criteria.

On the basis of this hypothesis theoretical formulas are obtained for the first critical heat flux in a large volume of liquid with free convection.

The qualitative conclusions from the hydrodynamic hypothesis are also confirmed well enough by the data on forced flow.

For further development of this problem it is necessary to carry on a number of widespread investigations of critical phenomena in the flow of liquids with different physical properties. It is also necessary to specify exactly the conditions of experiments.

It is especially necessary to accumulate experimental data in the region of positive flow vapour contents.

(3) The influence of circulation rate on the heat transfer coefficient with nucleate boiling depends on the ratio of intensities of heat transfer with conventional convection, and with developed boiling. In certain cases this influence may be quite essential and promote the operation of heat transfer equipment.

(4) The strict formulation of the problem of heat transfer intensity with bubble boiling is possible only on the basis of consideration of statistics of many final elements variable in space and time. At present such statistics are not available.

The ways of overcoming this difficulty discussed here in terms of deriving similarity conditions lead to an adequately complete system of hydrodynamic and thermal criteria. It is impossible to give strict formulation of the conditions on a heating surface.

(5) In practice the heat transfer conditions are not studied for boiling of binary mixtures or in other specific cases. The amount of work in this respect in no way corresponds to the practical and theoretical importance of this problem.

REFERENCES

1. E. K. Averin and G. N. Kruzhilin, *Izv. AN SSSR, OTN*, 10 (1955).
2. I. T. Aladiev, Konvektivnyi i luchistyi teploobmen, *Sbornik, Izdanie AN SSSR* (1960).

3. I. T. ALADIEV and L. D. DODONOV (see Ref. 2).
4. I. T. ALADIEV, V. E. DOROSHCHUK, Z. L. MIROPOLSKY and M. A. STYRIKOVICH, *Doklad na mezhdunarodnoi konferentsii po teploobmenu* (1961).
5. I. A. ANDREEV, *Trudy NII NKTP* 2–3 (1945).
6. V. M. BORISHANSKY, *Zh. Tekh. Fiz.* 2 (1956).
7. V. M. BORISHANSKY, *Energomashinostroenie*, 7, 5 (1958).
8. V. M. BORISHANSKY, Dostizheniya i zadachi v proizvodstve i primenenii kholoda, *Sbornik, LTIKhP* (1960).
9. Voprosy teploobmena pri izmenenii agregatnogo sostoyaniya veschestva, *Sbornik statei, GEI* (1953).
10. A. A. GUKHMAN, *Zh. Tekh. Fiz.* 6 (1953).
11. V. E. DOROSCHUK and F. P. FRID, Teploobmen pri vysokikh teplovykh nagruzkah, *Sbornik, GEI* (1959).
12. B. A. ZENKEVICH and V. I. SUBBOTIN, *Atomn. Energ.* 8 (1957).
13. B. A. ZENKEVICH, *Atomn. Energ.* 1, 74 (1958).
14. L. M. ZYSINA, Za novoe sovetskoe energooborudovanie, *Sbornik, OBTI TsKTI* (1939).
15. L. M. ZYSINA and S. S. KUTATELADZE, *Zh. Tekh. Fiz.* 1 (1950).
16. A. A. IVASHKEVICH, *Atomn. Energ.* 1 (1960).
17. Issledovaniya teplootdachi k paru i vode, *Sbornik statei, Atomizdat* (1958).
18. T. A. KOLACH, *Trudy MEI*, XXIV (1956).
19. M. V. KIRPICHEV, Teoriya podobiya, *Izd. AN SSSR* (1953).
20. M. A. KICHIGIN and N. Y. TOBILEVICH, Gidrodinamika i teploobmen pri kipenii, *Sbornik, Izd. AN SSSR* (1955).
21. G. N. KRUZHILIN, *Izv. AN SSSR, OTN* 7 (1948).
22. G. N. KRUZHILIN and V. I. SUBBOTIN, *Doklad No.* 2054 *na vtoroi Zhenevskoi konferentsii* (1958).
23. S. S. KUTATELADZE, *Teploperedacha pri izmenenii agregatnogo sostoyaniya veschestva*, Mashgiz (1939).
24. S. S. KUTATELADZE, *Teploperedacha pri kondensatsii i kipenii*, Mashgiz (1949, 1952).
25. S. S. KUTATELADZE and M. A. STYRIKOVICH, Gidravlika gazozhidkostnykh sistem, *GEI* (1958).
26. S. S. KUTATELADZE, Nauchnye doklady vysshei shkoly, *Energetika*, 2 (1959).
27. S. S. KUTATELADZE and B. A. BURAKOV, Teploperedacha i gidravlicheskoe soprotivlenie v dvukhfaznom potoke, *Sbornik, GEI* (1961).
28. S. S. KUTATELADZE and G. I. BOBROVICH, *Atomn. Energ.* 12 (1960).
29. S. S. KUTATELADZE, *Energomashinostroenie*, 1, 12 (1961).
30. V. G. LEVICH, Fiziko-khimicheskaya gidrodinamika, *GIFML* (1959).
31. D. A. LABUNTSOV, *Teploenergetika*, 12, 79 (1959); 5, 7 (1960).
32. Z. L. MIROPOLSKY, M. E. SHITSMAN, I. L. MOSTINSKY and A. A. STAVROVSKY, *Teploenergetika*, 1 (1959).
33. E. I. NESIS, *Zh. Tekh. Fiz.* 9 (1952).
34. I. I. NOVIKOV, Teploperedacha i gidravlicheskoe soprotivlenie v dvukhfaznom potoke, *Sbornik, GEI* (1961).
35. A. P. ORNATSKY, Issledovanie protsessov teploobmena pri bolshikh teplovykh potokakh, *Sbornik, Izd. AN USSR* (1955).
36. P. I. POVARNIN and S. T. SEMENOV, *Teploenergetika*, 4, 72 (1959); 1, 79 (1960).
37. V. D. POPOV, Teplovii rozrakhunok vakuum-aparativ, *DVTL* (1958).
38. V. A. ROBIN, *Teploenergetika*, 5, 61 (1958).
39. RAYLEIGH, Teoriya zvuka, *GIITL* (1955).
40. A. I. RYCHKOV, *Inzh.-Fiz. Zh.* 11 (1959).
41. A. F. SOROKIN, *Trudy MEI*, XXIV (1956).
42. L. S. STERMAN, *Zh. Tekh. Fiz.* 2 (1953).
43. L. S. STERMAN and N. G. STYUSHIN, *Trudy TsKTI*, kniga 21 (1951).
44. L. S. STERMAN, N. G. MOROZOV and S. L. KOVALEV, *Inzh.-Fiz. Zh.* 5 (1959).
45. M. A. STYRIKOVICH and L. E. FAKTOROVICH, *Teploenergetika*, 2, 83 (1959).
46. M. A. STYRIKOVICH, Z. L. MIROPOLSKY, M. E. SHITSMAN, I. L. STAVROVSKY and L. E. FAKTOROVICH, *Teploenergetika*, 5, 83 (1960).
47. N. V. TARASOVA, A. A. ARMAND and A. S. KON'KOV, Teploobmen pri vysokikh teplovykh nagruzkakh, *Sbornik, GEI* (1959).
48. G. G. TRESHCHEV (see Ref. 47).
49. N. Y. TOBILEVICH, *Trudy TsINS*, V (1958).
50. V. I. TOLUBINSKY, Izv. vysshikh uchebnykh zavedenii, *Energetika*, I, 15 (1959).
51. V. G. FASTOVSKY and R. U. ARTYM, *Teploenergetika*, 8, 74 (1958).
52. V. S. CHIRKIN and V. P. YUKIN, *Zh. Tekh. Fiz.* 7 (1956).
53. L. S. EIGENSON, *Modelirovanie, Izd. Sov. Nauka* (1952).
54. B. J. BANKOFF, *J. Chem. Engng*, 4 (1958).
55. F. BOŠNJAKOVIĆ, *Tech. Mech. Therm.* 1 (1930).
56. H. BUCHBERG, *Heat Transfer and Fluid Mech. Inst.*, June (1951).
57. C. COURTY and A. FOUST, *Chem. Engng Progr. Symp.*, ser. 51 (1955).
58. H. CICHELLI and C. BONILLA, *Trans. Amer. Inst. Chem. Engng*, Dec. (1945).
59. H. FORSTER and N. ZUBER, *Appl. Phys.* 25 (1954).
60. W. FRITZ, *Z. Phys.* 36 (1935).
61. HADAMARD, *C.R. Acad. Sci., Paris* (1952).
62. S. LEVY, *J. Heat Transfer*, 2 (1959).
63. M. JACOB and W. FRITZ, *Forschung*, 2 Bd, Heft 12 (1931).
64. M. JACOB and LINKEW, *Z. Phys.* 36 (1935).
65. M. JACOB, *Heat Transfer*, No. 9 (1949).
66. M. PLESSET and S. ZWICK, *J. Appl. Phys.* 25 (1954).
67. W. ROHSENOW, *ASME Trans.* 74 (1952).
68. VAN W. WIJK, A. VOS and VAN S. STRALEN, *Chem. Engng Sci.* 5 (1956).
69. VAN S. STRALEN, *Warmteokerdracht aan kokende binaire veocistofmengsels*, Wegeningen (1959).
70. N. ZUBER, *Trans. ASME* 3 (1958).
71. N. ZUBER and M. TRIBUS, *Dept. of Engng, Univ. of California, Rep.* 58–5 (1958).
72. N. ZUBER, Hydrodynamic aspects of boiling heat transfer (thesis), *AECU*-4439 (1959).

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Abstract—The present paper is devoted to the problem of nucleate boiling heat transfer.

Complicated hydrodynamic processes occurring in gas–liquid systems with nucleate boiling have made it impossible up to now to develop a complete theory for this process.

A system of differential equations for a nucleate boiling process has been formulated and a rational system of criteria characterizing it has been developed, proceeding from the hypothesis on the hydrodynamic nature of crises in the boiling mechanism.

Generalization of experimental works on nucleate boiling has been carried out on the basis of the analysis of some principal problems on the boiling heat transfer theory and a system of hydrodynamic and heat criteria characterizing this process.

Résumé—Cet article est consacré au problème de la transmission de chaleur au cours d'une ébullition nucléée.

Les phénomènes hydrodynamiques compliqués qui interviennent dans le système liquide–gaz lors d'une ébullition nucléée ont jusqu'à présent rendu impossible, l'établissement d'une théorie complète de ce processus.

Cet article établit un système d'équations différentielles relatif à une ébullition nucléée caractérisée par des critères rationnels fondés sùr l'hypothèse faite sur la nature hydrodynamique du mécanisme de l'ébullition.

La généralisation des travaux expérimentaux sur l'ébullition nucléée a été faite à partir de l'étude de quelques problèmes essentiels de la théorie de la transmission de chaleur par ébullition et d'un système de critères thermiques et hydrodynamiques caractéristiques de ce processus.

Zusammenfassung—Der Wärmeübergang beim Blasensieden ist hier behandelt. Die komplizierten hydrodynamischen Prozesse, die in Gas–Flüssigkeit-Systemen beim Blasensieden auftreten, machten es bis jetzt unmöglich, eine geschlossene Theorie für diesen Vorgang zu entwickeln. Hier wird ein System von Differentialgleichungen für den erwähnten Vorgang aufgestellt und durch eine Reihe von Kriterien charakterisiert, die auf der Annahme der hydrodynamischen Natur der burn-out-Erscheinungen beruhen. Auf Grund der Analyse einiger prinzipieller Probleme des Wärmeübergangs beim Sieden und aus einem System hydrodynamischer und wärmetechnischer Charakteristika des Vorgangs konnten Verallgemeinerungen für experimentelle Arbeiten getroffen werden.

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HEAT AND MASS TRANSFER BETWEEN THE GASEOUS AND LIQUID PHASES OF A BINARY MIXTURE

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Abstract—A rigorous theoretical analysis is presented of the heat and mass transfer occurring at an element of the interface separating the liquid and gaseous phases of a binary mixture. Solutions of the equations are presented for ethanol–water mixtures, three separate cases being considered, namely—(i) the bulk liquid and gas states are not saturated, and both Lewis numbers equal to unity; (ii) the bulk states are saturated and both Lewis numbers equal unity; and (iii) the liquid Lewis number is close to zero, while the gaseous one equals unity, the bulk states being either saturated or not. Results are compared with the predictions of the approximate (equi-molal transfer) treatment conventionally used (e.g. [1]).

It is shown that the conventional theory only gives tolerable accuracy for case (ii); in the other cases its prediction bears almost no relation to those of the rigorous theory.

The rigorous theory shows that, in the examples considered, either net condensation or net vaporization may occur, depending on the conductance values and the bulk states. In some cases the more volatile component is caused to condense despite having a higher gas-phase concentration at the interface than in the bulk of the gas.

NOMENCLATURE

B, driving force for mass transfer (equation 1) (Dimensionless);

f, mass of one of the components per mass of mixture (equation 3) (Dimensionless);

g, mass-transfer conductance (equation 1) (kg/m²h);

g^*, value of g corresponding to zero mass transfer (equation 9) (kg/m²h);

h, specific enthalpy of mixture (equation 4) (kcal/kg);

Δh_i, latent heat of vaporization of component i (equation 16a) (kcal/kg);

$\dot{m}''$, net mass transfer rate (equation 1) (kg/m²h);

m_i'', mass transfer rate of component i (equation 19) (kg/m²h);

M_i, molecular weight of component i (equation 16) (kg/kg mole);

P, any conserved property (equation 2) (various);

t, temperature (equation 10) (°C).

Subscripts

f, h, pertaining respectively to concentration and to enthalpy;

I, II, respectively the gaseous and the liquid phases;

i, C_2H_5OH, H_2O, Mixture components;

G, S, T, L, F, G_S, T_S, F_L, T_L, Mixture states.

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1. INTRODUCTION

1.1. *The problem considered*

We shall consider the interactions which occur across the interface between a liquid and a gaseous phase, when each consists of a mixture of the same two chemical compounds. Such liquid–gas contact of binary mixtures occurs in many situations of engineering importance, for example: in the rectification column of a plant producing liquid air; in the de-aerator of a steam power plant; and in the rectification column of an absorption-type refrigeration plant employing ammonia–water mixtures.

The questions to be asked are: How can we predict the rates of heat and mass transfer across an element of the interface when the bulk states of the liquid and of the gas are given? What is the nature of the material which is transferred? And how are the answers to these questions dependent on the "conductances" of the liquid and gas boundary layers?

Answers to the questions are needed for three

reasons. Firstly, a knowledge of the answers might facilitate the design of continuous-contact rectification columns in the future. Secondly, such knowledge throws light on the transfer processes which occur between the vapour bubbles and the liquid on the trays of a plate-type column. Thirdly, although many of the most important rectification processes, for example those occurring in the petroleum industry, involve multicomponent mixtures, it seems certain that a full understanding of the processes in binary mixtures must precede that of the more general case.

It should be stressed that attention is confined to local and instantaneous conditions: we shall not here consider how our findings influence the variations with position of the bulk liquid and gas states throughout the tower or the variations with time of these states in an unsteady-state experiment. However, the present study is a necessary preliminary to the solution of the latter problems, which it is hoped to deal with in later publications.

1.2. *Earlier treatments of the problem*

Despite the crucial importance of the process under consideration, no rigorous treatments of it appear to have been published,† at least in the Western literature. Instead, textbooks on mass transfer, distillation, absorption, etc. [1–3] present an approximate treatment similar to that described in section 3 below.

However, the approximate treatment, though satisfactory in many cases, sometimes predicts mass transfer rates which are incorrect, not only in magnitude but also in sign. Moreover, it obscures some aspects of the phenomenon which must be clearly perceived if understanding of the efficiency of bubble-cap plates, for example, is to advance.

1.3. *Purposes of present work*

The aim of the present paper is therefore to provide a rigorous treatment of the problem, i.e. to provide, for the questions indicated above, answers which are consistent with the laws of heat and mass transfer. This purpose could be fulfilled simply by writing down the complete set of governing equations and indicating a suitable numerical solution procedure; the more extended discussion presented below must therefore be justified by reference to two further aims.

The first of these is to provide *insight* into what governs the heat and mass transfer processes; it is for this purpose that graphical methods are employed. The second aim is to see whether these processes can be adequately described and calculated by fitting them into the framework of a recently developed standard formulation of the convective mass-transfer problem [4].

2. THEORY

2.1. *The system considered*

Figure 1 illustrates an element of the interface separating the gas from the liquid. It will be supposed that gradients of composition and temperature are confined to relatively thin boundary layers on either side of the interface, and that outside these regions it is meaningful to speak of the "bulk" states of the gas and of the liquid. These will be designated the G- and F-states respectively.

The state of the gas immediately adjacent the interface is generally different in both composition and temperature from that of the bulk of

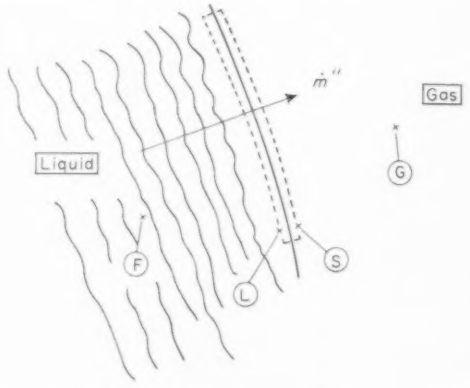

FIG. 1. Illustrating an element of the interface between a liquid and a gaseous phase.

† The most complete treatment known to the writer is that of Colburn and Drew [6]. These authors were probably aware of all the qualitative features of binary mixture interphase transfer which are described in the present report. However, their paper paid most attention to differential condensation on a dry wall and did not explore the general problem treated here.

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the gas; it will here be called the S-state. The liquid immediately adjacent the interface can be taken as in thermodynamic equilibrium with the gas in state S; this liquid state will be denoted by the letter L.

In addition to the G-, F-, S- and L-states, one further state will require consideration; this is the "transferred-substance" or T-state. However, this state is not normally *possessed by* the mixture at any particular location; discussion of its significance will therefore be deferred.

Practical relevance. Figures 2(a) and 2(b) illustrate the relevance of the elementary situation of Fig. 1 to practical equipment. In the

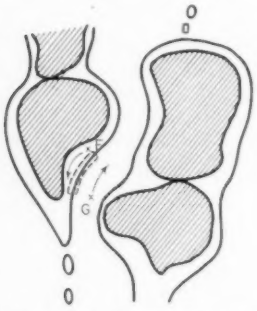

FIG. 2(a). Illustrating the interface element, and the F and G streams, in the context of packed-bed transfer equipment.

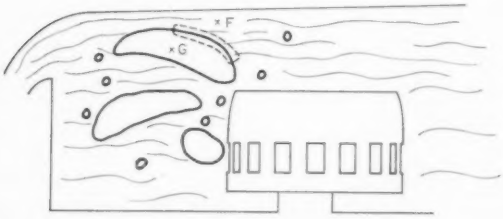

FIG. 2(b). Illustrating the interface element, and the F and G streams in the context of bubble-cap-tray transfer equipment.

former, the interface element lies between a film of liquid, running over the surface of an irregularly shaped solid in a packed tower, and a stream of gas which flows upward through the interstices of the packing. In Fig. 2(b), the G-state is the instantaneous average state of a gas bubble which rises up through liquid which flows towards the downspout of a bubble-cap plate.

Thermodynamic description. It will be supposed that only two chemical substances are present, and that these may mix in any proportions in both the gaseous and the liquid phases. For example, the two materials might be water and ethanol (C_2H_5OH), the thermodynamic properties of which, for a total pressure of 1 atm, are represented in Fig. 3 [5]. The latter is a diagram with enthalpy as ordinate and mass fraction as abscissa; such diagrams will prove to be extremely helpful in the following discussion.

As an example, four state-points have been marked on Fig. 3: G, F, S and L. These correspond to four states discussed above. It will be noticed that S and L each lie on their respective saturation line, at opposite ends of a mixed-phase isotherm (92°C). The G and F points have been placed in typical positions in relation to the L and S points.

In most problems, the G- and F-states are given, whereas the L- and S-states are not; we shall shortly have to consider ways of determining the locations of points L and S.

2.2. *Equations*

The "Ohm's Law" equations. In the standard formulation referred to above [4], the net rate of mass transfer into the phase under consideration, $\dot{m}''$ (kg/m²h), is expressed by:

$$m'' = g \cdot B \tag{1}$$

where g = a "conductance", mainly determined by aerodynamic factors (kg/m²h), and B = a "driving force" (dimensionless), determined by the values of a conserved property P in the G-, S- and T-states, usually in accordance with:

$$B \equiv \frac{P_G - P_S}{P_S - P_T} \tag{2}$$

wherein the subscript denotes the state referred to.

We shall use the same formulation here, with the following additional distinctions:

(ii) The vapour phase will be designated I, and the liquid phase II. The conductances g_I and g_{II} consequently refer to the gaseous and the liquid boundary layers respectively.

(ii) The conserved properties which will be

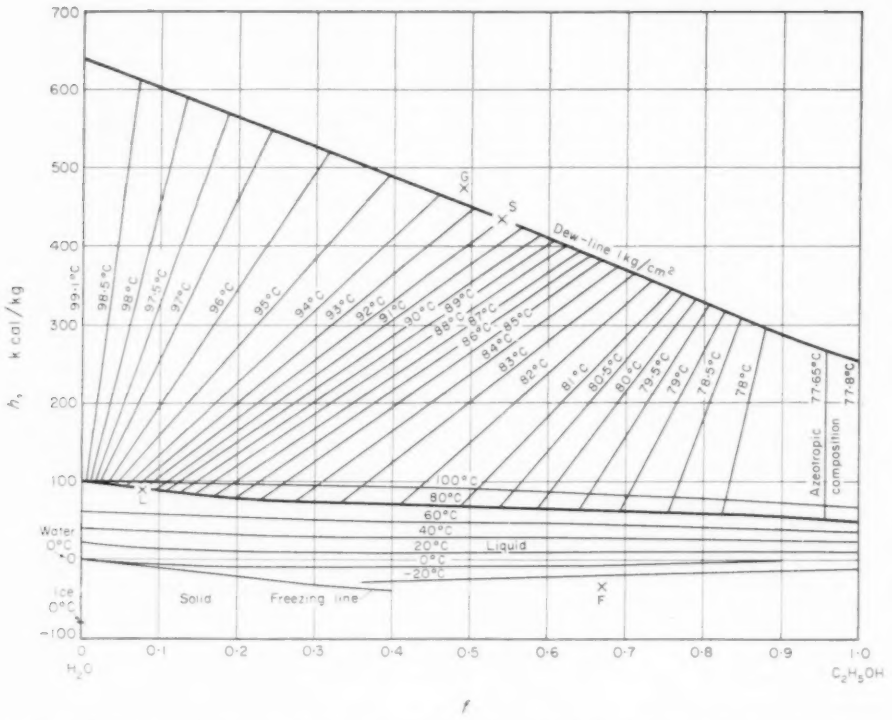

FIG. 3. Enthalpy–composition (h–f) diagram for mixtures of ethyl-alcohol (C_2H_5OH) and water at 1 atm pressure [5].

used are h and f, respectively the ordinate and abscissa of diagrams such as Fig. 3.

h stands for the specific enthalpy of the mixture (kcal/kg), while f stands for the mass of one component of the mixture (ethanol in Fig. 3) per unit mass of mixture.

We may therefore distinguish two forms of the driving force appropriate to the gas phase, based respectively on f and on h. These are:

$$B_{I,f} \equiv \frac{f_G - f_S}{f_S - f_T} \tag{3}$$

$$B_{I,h} \equiv \frac{h_G - h_{G_S}}{h_{T_S} - h_T} \tag{4}$$

wherein state G_S is that of material having the composition of G but the temperature, phase and state of molecular aggregation of S, while state T_S similarly has the composition of T and the temperature, phase and state of molecular aggregation of S.

Corresponding equations may be written for the liquid phase, introducing the states F_L and T_L, defined in a similar fashion. All the states† referred to are represented by state-points in Fig. 4, wherein the broken lines G_SST_S and LF_LT_L are prolongations of pure-phase isotherms.

(iii) Since the exchange coefficients governing

† Equation (4) has the same form as equation (2) provided that the convention is adopted that all substances having the temperature, phase and state of aggregation of S have zero enthalpy. Since it is often inconvenient to adopt this convention, equation (4) should be used; its use corresponds to case (vi) of section 2.3 of [4].

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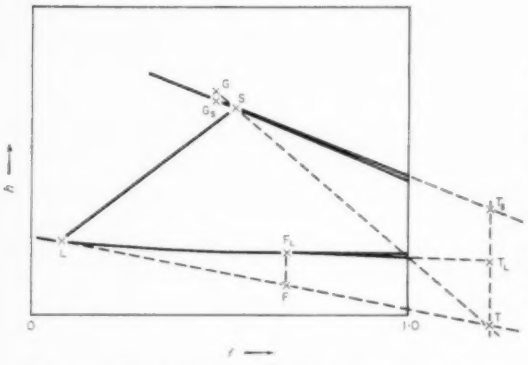

FIG. 4. Illustrating the locations of the state-points G_S, T_S, T and T_L. The diagram is drawn for the case in which GST and FLT are straight lines. (Lewis numbers equal to unity.)

the diffusion of matter and heat (respectively the diffusion coefficient times the mixture density, and the mixture thermal conductivity divided by the mixture specific heat at constant pressure) normally have different values, the conductances appropriate to B_f and B_h will differ. We therefore distinguish the two conductances for the gas phase: $g_{I,f}$ and $g_{I,h}$.

(iv) In the standard formulation of [4], it was possible to focus attention on one of the two phases, namely that for which G was the bulk state. Now that two phases are present, the phase containing mixtures F and L is of equal importance. However, to avoid introduction of further notation, we allow a certain asymmetry to enter, by retaining the convention that $\dot{m}''$ is positive when the net mass transfer is towards the phase designated G.

We are now in a position to write down the four "Ohm's Law" equations, corresponding to the four "resistances" to transfer, two of which reside in phase I (gas) and two in phase II (liquid). The equations run:

$$\dot{m}'' = g_{I,f}(f_G - f_S)/(f_S - f_T) \quad (5)$$

$$= g_{I,h}(h_G - h_{G_S})/(h_{T_S} - h_T) \quad (6)$$

$$= -g_{II,f}(f_F - f_L)/(f_L - f_T) \quad (7)$$

$$= -g_{II,h}(h_F - h_{F_L})/(h_{T_L} - h_T). \quad (8)$$

How the conductances are determined. It will be assumed that sufficient information is available for the values of the g's to be determined; for example, knowledge of the liquid-film thickness and velocity and of the appropriate exchange coefficients could lead to a determination of the g_{II}'s. More precisely, such knowledge could enable a g^* to be calculated, this being the value of the g which prevails as B tends to zero. g and g^* are related to B by an equation which has the general characteristics of:

$$\frac{g}{g^*} = \frac{\ln(1+B)}{B} \quad (9)$$

to the symbols of which the appropriate I, II, h and f subscripts can be added; the exact form of the relation depends on the flow conditions [4]. For the present, it will be helpful to regard the g's as easily determined quantities, and to confine attention to the $g(B)$ relation of equation (9).

Thermodynamic-property relations. In addition to equations (5), (6), (7) and (8), relations exist between the S and L properties. These express, among other things, the requirement of thermodynamic equilibrium at the interface and may be written in terms of the interface temperature t_S (which is equal to t_L) as:

$$f_S = f_S(t_S) \quad (10)$$

$$h_{G_S} = h_{G_S}(t_S, f_G) \quad (11)$$

$$h_{TS} = h_{TS}(t_S, f_T) \quad (12)$$

$$f_L = f_L(t_S) \quad (13)$$

$$h_{F_L} = h_{F_L}(t_S, f_F) \quad (14)$$

$$h_{T_L} = h_{T_S}(t_S, f_{T_S}). \quad (15)$$

Here the bracket denotes " is a function of ".

The transferred-substance state T. It is necessary to refer to [4] for a full discussion of the significance of this state. Here it may suffice to state:

(i) That the T-state is the state which a single fluid stream must possess if, in flowing from the F to the G phase at a mass rate $\dot{m}''$, it is to have the same effects as the totality of mass and heat fluxes which

actually cross control surfaces at either the S or the L locations.

(ii) The T-state can be represented by a state-point on an enthalpy–composition diagram, but this point is not constrained to lie between the lines $f = 0$ and $f = 1$. Fig. 4 in fact illustrates the situation in which f_T is greater than unity.†

2.3. *The mathematical problem*

The data. As has been indicated already, the mathematical problem is specified in terms of given bulk-fluid states and given conductances; thus all the g's and h_G, f_G, h_F and f_F, can be regarded as known.

The required information. We normally require to find out the net mass transfer rates, $\dot{m}''$, and the composition of the transferred substance, e.g. f_T. The latter quantity gives the rate of transfer of the substance whose mass fraction is measured by f: for this is equal to $\dot{m}''f_T$. The transfer rate of the other component is of course $\dot{m}''(1 - f_T)$.

Equations and unknowns. The equations to be solved are (5)–(8) and (10)–(15); there are ten of them. There are also ten unknowns, namely: f_S, f_T, f_L, h_{G_S}, h_{T_S}, h_T, h_{F_L}, h_{T_L}, t_S and $\dot{m}''$. The problem is evidently soluble.

Since the equations are non-linear, and since the relations contained in equations (10)–(15) may not be expressible through simple functions, trial-and-error procedures will ordinarily be necessary. However there is no need to discuss such prosaic matters here.

Discussion of the problem in graphical terms. It is helpful to interpret the mathematical problem by reference to Fig. 4. Two facts stand out: the locations of points G and F are known; and the locations of T, S and L are sought. With the latter points located, G_S, T_S, F_L and T_L can be found immediately; indeed, since S and L are tied together, only one of these need be regarded as an independent unknown. Graphical representation of the problem on an enthalpy–composition chart thus suggests that attention should be concentrated at first on establishing the interface temperature t_S (hence S and L) and the transferred-substance state T; with these known, everything else follows quickly.

† In the work of Colburn and Drew [6], f_T is represented (in molal units) by the symbol z. The authors made it clear that z could lie outside the region between 0 and 1.

3. THE CONVENTIONAL APPROXIMATE PROCEDURE

3.1. *Simplification of the mathematical problem*

The basic assumption. The mathematical problem displayed in the last section is rarely solved rigorously; at any rate, no rigorous solutions appear to have been published. Instead, the problem is usually simplified by making an assumption about the transferred substance. This assumption contains the following elements:

(i) The rate of vaporization of the more volatile compound bears a known ratio to the rate of condensation of the less volatile component. In the present terms, this means: f_T is known.

(ii) The above ratio is equal to the inverse ratio of the latent heats of vaporization of the respective pure components. This implies that the mixtures are ideal.

(iii) *The molal* latent heats of vaporization are assumed to be equal.

(iv) Consequently *the molal condensation rate of the less volatile component is equal to the molal vaporization rate of the more volatile component.* The last statement is the basic assumption referred to in the title.‡ It may be expressed symbolically as:

$$f_T = 1/(1 - M_j/M_i) \qquad (16)$$

where

M_i = molecular weight of the substance i, the one whose concentration is measured by f, and

M_j = molecular weight of the other component of the mixture.

‡ Of course there is no need to make step (iii), which involves a further approximation; instead, equation (16) could be replaced by the more correct equation:

$$f_T = 1/(1 - \Delta h_i/\Delta h_j) \qquad (16a)$$

where Δh is the latent heat of vaporization. Something of this kind is done when "fictitious molecular weights" [3] are employed; the direct use of equation (16a) is less devious and renders the mention of molecular weights entirely unnecessary.

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Consequences. The main consequence of equation (16) is that it permits further considerations of the equations involving h to be dispensed with. The remaining equations are:

$$\dot{m}'' = g_{\mathrm{I},f}(f_G - f_S)/(f_S - f_T) \quad (5)$$

$$\dot{m}'' = -g_{\mathrm{II},f}(f_F - f_L)/(f_L - f_T) \quad (7)$$

and

$$f_S = f_S(f_L) \quad (20)$$

the last equation resulting from the elimination of t_S between (10) and (13).

In these three equations there are only three unknowns, viz. f_S, f_L and $\dot{m}''$; for f_T is given by (16). The problem is therefore soluble; it is moreover considerably simpler than the full problem outlined in section 2.3.

3.2. *Method of solution*

Despite the simplicity of the reduced mathematical problem, graphical methods are often employed. Fig. 5 illustrates the type of diagram

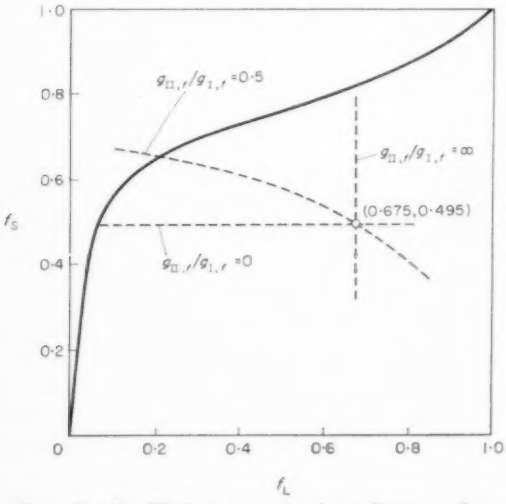

FIG. 5. Equilibrium-concentration diagram for mixtures of ethyl-alcohol and water at 1 atm pressure, showing construction lines for the example shown in the text, according to equation (21).

used, with f_S as ordinate and f_L as abscissa. The full curve represents the equilibrium relation (20), while the broken curve represents the equation resulting from the elimination of $\dot{m}''$ between equations (5) and (7), namely:

$$f_S - f_G = \frac{(f_L - f_F)(f_T - f_G)\, g_{II,f}/g_{I,f}}{(f_L - f_F)(1 + g_{II,f}/g_{I,f}) + (f_F - f_T)}. \quad (21)$$

When $g_{II,f}/g_{I,f} = 0$, this equation represents a horizontal straight line on Fig. 5; when $g_{II,f}/g_{I,f} = \infty$ the equation represents a vertical straight line; for all other values of $g_{II,f}/g_{I,f}$, the equation represents a curve unless f_T happens to be infinite (equal molecular weights), in which case it represents a line of slope $-g_{II,f}/g_{I,f}$.

If *molal* concentrations are used, in place of f, for Fig. 5, equation (21) is always represented by a straight line.

The values of f_S and f_L can be determined from the intersection of the equilibrium line in Fig. 5 with the curve representing equation (21) for a particular value of the conductance ratio $g_{II,f}/g_{I,f}$; thereafter $\dot{m}''$ is easily evaluated from equation (5) or (7). The value of the mass transfer rate of a particular component, say $\dot{m}_i''$, is subsequently found from the definition of f_T, namely:

$$\dot{m}_i'' = f_T \dot{m}''. \quad (22)$$

Numerical example. The above procedure has been applied to the ethanol–water example illustrated by Fig. 3, for which $f_G = 0{\cdot}495$ and $f_F = 0{\cdot}675$. Since it is conventional to ignore the influence of the mass-transfer rate on the conductance, g_f and g_f^* have been taken to be identical; inspection of equation (9) shows that this amounts to assuming that B is close to zero. The molecular weights are such that $f_T = 1{\cdot}643$.

Figure 6 represents the results of the calculation in the form of curves of the quantities t_S, $\dot{m}''_{C_2H_5OH}/g^*_{I,f}$, $\dot{m}''/g^*_{I,f}$ and $\dot{m}''_{H_2O}/g^*_{I,f}$. The abscissa quantity is $g^*_{I,f}/(g^*_{I,f} + g^*_{II,f})$, this being chosen in preference to the conductance ratio because it represents all physically realisable conditions in the finite range 0 to 1; the sum $(g^*_{I,f} + g^*_{II,f})$ has no particular physical significance.

Inspection of the curves of Fig. 6 shows that the process is one of net vaporization; for $\dot{m}''$ is always positive. However the rate of vaporization of ethanol is even greater than $\dot{m}''$; water on the other hand condenses, for $\dot{m}''_{H_2O}$ is negative

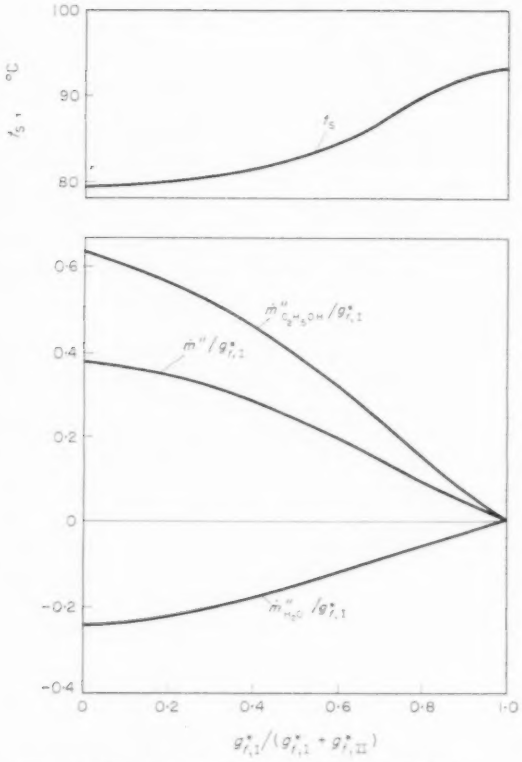

FIG. 6. Results of the approximate calculation method when applied to the example in section 3.2 of the text.

These results are qualitatively in accordance with conventional expectations and are an expression of the rectifying action of the interphase contact.

The shapes of the curves of Fig. 6 are such that the transfer rates are proportional to $g^*_{I,f}$ when this quantity is small (gas-phase control); when $g^*_{I,f}$ is much larger than $g^*_{II,f}$ however, the transfer rates depend only on the liquid-phase conductance. These results are also well known.

The curves of Fig. 6 have been presented primarily for comparison with the curves expressing the results of more rigorous theoretical predictions (Figs. 8, 10 and 12).

3.3. *Discussion*

The calculation method just described is easy to perform, and provides definite answers to most of the questions which interest the equipment designer. Why then should we look further?

There are several answers to this question. Firstly, as will be shown below, the predictions of the conventional theory, though definite and easy to obtain, are often entirely incorrect; this at any rate is the conclusion which follows from the analysis given below, although it must be admitted that experimental data which bear on the matter are hard to come by. Secondly, the procedure described is quite different from that which is used in, for example, air-conditioning or cooling-tower design; yet the water–air system is merely a particularly simple binary-mixture system, so should be amenable to the present treatment. Thirdly, the disappearance of the enthalpies from the calculation occasions justifiable misgivings: the conventional theory predicts the same rates of material transfer, for fixed gas and liquid compositions, regardless of whether the gas is superheated; can this be correct?

In the following sections a rigorous treatment of the problem will be presented: here we shall no longer *assume* the value of f_T in the manner of section 3.1, but shall solve the general problem described in section 2.3. Three examples will be studied; these are chosen so as to have the same gas and liquid bulk compositions as in the example just considered; comparison of their solutions with that displayed in Fig. 6 will therefore enable the correctness of the conventional method to be assessed.

4. RIGOROUS SOLUTION OF THE MATHEMATICAL PROBLEM WHEN THE LEWIS NUMBERS ARE UNITY

4.1. *The implications of the Lewis number restriction*

Definition. The Lewis number is here understood to be the material exchange coefficient (i.e. diffusion coefficient times density) divided by the thermal exchange coefficient (thermal conductivity divided by constant-pressure specific heat). Its value is usually close to unity in gases and in all turbulent fluids; it is however much less than unity in most non-turbulent liquids. In the present section we consider the case in which the Lewis number is equal to

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unity in both the liquid and the gaseous phase.

Influence on conductances. It is shown in [4] (and, of course, in many other places as well) that when the Lewis number is unity for a phase, the g_h and g_f conductances for the phase in question are equal.

Influence on the location of the state points. Inspection of equations (5) and (6), for example, together with interpretation of the f- and h-differences appearing there as lengths on Fig. 4, quickly reveals that, if $g_{I,f}$ and $g_{I,h}$ are equal, the state-points G, S and T must lie on a straight line. For similar reasons, F, L and T are also co-linear.

Incidentally, in this case it is permissible to write the driving force $B_{I,h}$, for example, as $(h_G - h_S)/(h_S - h_T)$; this is in accord with simplification (ii) to the differential equations of [4].

The co-linearities of G, S and T and of F, L and T ensure that there is only one state-point on the diagram h–f which has to be established: for with G and F fixed, specification of the location of any one of F, L or T immediately fixed the other two. Fig. 4 illustrates this.

4.2. *Possible and impossible conditions*

The length ratios representing B_I and B_{II}. It is evident that if g_f and g_h are equal, we may drop the h and f subscripts from the B's, as well as from the g's. Further, the B's can be represented by length ratios on an h–f diagram, by reason of equations (5)–(8); we have:

$$B_I = \overline{GS}/\overline{ST} \tag{23}$$

and

$$B_{II} = \overline{FL}/\overline{LT} \tag{24}$$

where $\overline{GS}$, $\overline{ST}$ etc. stand for the lengths of the corresponding lines on the h–f diagram.

Now not all possible pairs of values of B_I and B_{II} correspond to physical reality. There are two restrictions: firstly, no B value can be less than -1, for this would contravene the Second Law of Thermodynamics; secondly, when B_I is positive, B_{II} is negative for reasons of continuity. It follows that the points S and L cannot range at will along the saturation lines on the h–f chart.

Permissible locations for S, L *and* T. The above reflections lead to the conclusion that S, L and T may only lie on the loci indicated by heavy lines in Fig. 7. Here the S and L loci are restricted by

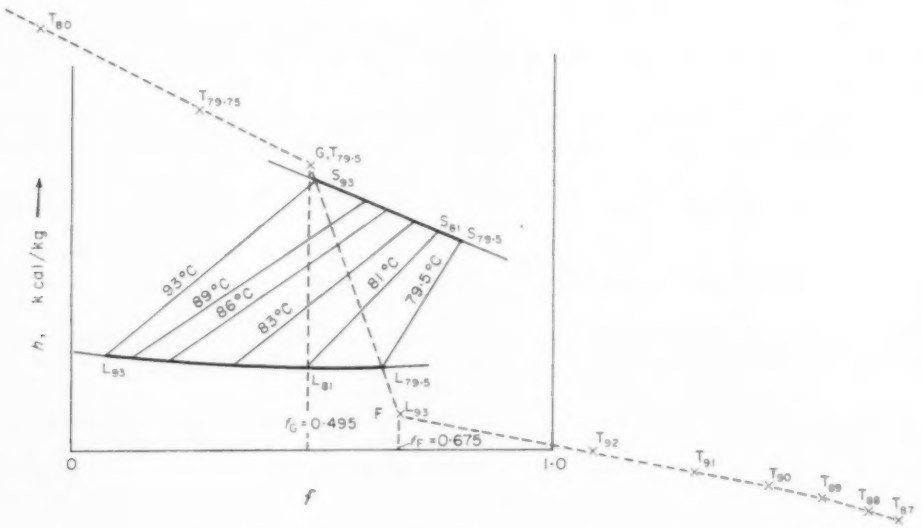

FIG. 7. Illustrating the permissible loci of S, L and T when G and F are fixed. The positions of G and F, and the shape of the h–f diagram, correspond to the C_2H_5OH–water example discussed in section 4 of the text.

the mixed-phase isotherms which pass through the intersections of the saturation curves with the straight line GF; the T loci can be deduced from this requirement. If for example the point S were to lie elsewhere on its saturation line, it can be easily verified that one of the restrictions on B would be contravened.

The lengths of the permissible loci of S and L clearly depend greatly on the positions of G and F: if f_G is close to zero and f_F is close to unity, for example, S and L each has the freedom of almost the whole length of its saturation curve; on the other hand, if the line GF happened to coincide with a mixed-phase isotherm, only one pair of surface states would be possible.

4.3. *Determination of the actual interface conditions*

The equation. We have seen that S and L, and so t_S, can only vary within certain limits; we now determine where within these limits they actually lie in a given case. To do this we equate two quantities derived from equations (5) to (8), equation (9) and equations (23) and (24); there results:

$$g_{II}^*/g_I^* = -\frac{\ln(1 + \overline{GS}/\overline{ST})}{\ln(1 + \overline{FL}/\overline{LT})} \qquad (25)$$

wherein the h- and f-subscripts have been omitted as redundant.

The role of the conductance ratio. Now $\overline{GS}/\overline{ST}$ and $\overline{FL}/\overline{LT}$ are functions of t_S alone, once G and F are fixed, as has been seen. It is therefore possible to evaluate for every value of t_S, the quantity:

$$-\ln(1 + \overline{GS}/\overline{ST})/\ln(1 + \overline{FL}/\overline{LT});$$

according to equation (25) this is equal to the ratio of the zero-mass-transfer conductances of the two phases, i.e. g_{II}^*/g_I^*.

The influence of the conductance ratio is brought out by the top curve of Fig. 8 which has been derived in the way just indicated for the situation illustrated in Fig. 7. This shows the interface temperature, t_S, plotted versus the quantity $g_I^*/(g_I^* + g_{II}^*)$, in the same manner as for Fig. 6. We see that the interface temperature is highest, in the present case, when the gas-side

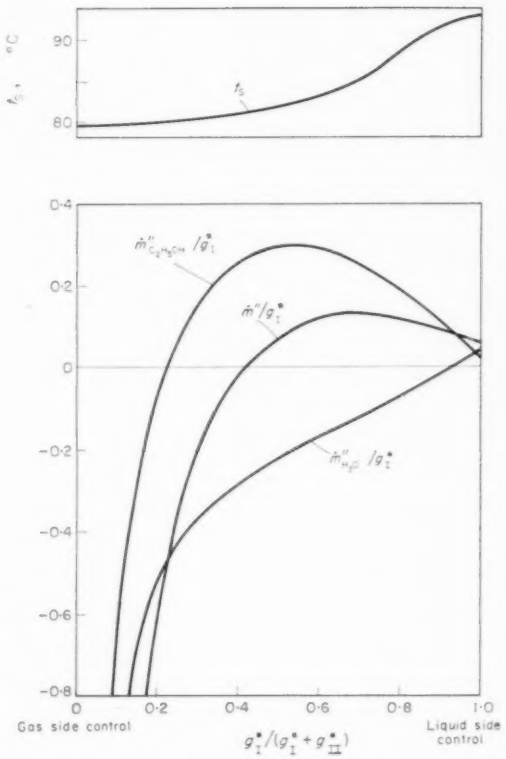

FIG. 8. Results of the rigorous calculation indicated in Fig. 7 (G superheated and F supercooled).

conductance is the larger; low interface temperatures are associated with large liquid-side conductances.

Comparison with the approximate treatment. Qualitatively, the interface-temperature variation just described accords with that predicted by the approximate treatment of section 3. Comparison of the t_S curves of Figs. 6 and 8 reveals slight quantitative differences between the results of the approximate and of the rigorous treatments, but these are of little account in the present case. It is easy to see that the differences would be larger if the G and F states of our problem lay further from their respective saturation lines (highly superheated vapour, highly sub-cooled liquid).

4.4. *The mass transfer rates*

Net mass transfer rate. Now that the interface

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condition has been established, we may evaluate the mass transfer rate $\dot{m}''$, for given values of the conductances by way of equations (5) to (8). Fig. 8 exhibits a curve containing this information valid for the example specified in Fig. 7; it is the curve with co-ordinate $\dot{m}''/g_{\mathrm{I}}^*$. Inspection of this curve shows that $\dot{m}''$ is positive when g_{I}^* is greater than $(0{\cdot}42/0{\cdot}58)g_{\mathrm{II}}^*$; then the process is one of net vaporization. When g_{II}^* is larger than $(0{\cdot}58/0{\cdot}42)g_{\mathrm{I}}^*$ on the other hand, $\dot{m}''$ is negative, signifying that there is a net condensation. As is easy to understand, vaporization thus occurs when the interface temperature is high, and condensation when it is low.

Comparison of the $\dot{m}''$ curve of Fig. 8 with that of Fig. 6 shows some startling differences. The conventional theory is correct (i.e. Fig. 6 gives the same value of $\dot{m}''/g_{\mathrm{I},f}^*$ as in Fig. 8) only for a value of the abscissa around 0·73; when the abscissa quantity is less than 0·42, however, the conventional theory is wrong even as to the sign of the net mass-transfer rate.

Ethanol vaporization rate. The curve for $\dot{m}''_{\mathrm{C_2H_5OH}}/g_{\mathrm{I},f}^*$ shown in Fig. 8 has been obtained by multiplying values of $\dot{m}''/g_{f,\mathrm{I}}^*$ by f_{T}, which is of course the abscissa of T on Fig. 7; it lies above that for the net mass transfer rate over almost the whole of the conductance range; in this respect it accords with the curve obtained from the conventional theory, in Fig. 6. The curves scarcely agree in any other respect however: the rigorous theory shows that even ethanol condenses when the gas-side conductance is low enough.

It is worth noting that the ratio $\dot{m}''_{\mathrm{C_2H_5OH}}/\dot{m}''$, i.e. f_{T}, is shown by Fig. 7 to be capable of taking any value, positive or negative, outside the range from 0·495 to 0·695. It has the value assumed in the conventional theory (namely 1·643, given by evaluating equation (16) with $M_i = 46$ and $M_j = 18$) for only one value of the interface temperature, namely 85·8°C. The corresponding value of $g_{\mathrm{I},f}^*/(g_{\mathrm{I},f}^* + g_{\mathrm{II},f}^*)$ is 0·67; this is of course near† the condition, noted above, at which the net mass-transfer rates predicted by the conventional and the exact theories are equal.

† The agreement would be precise did not the conventional theory also include the approximation: $g_f = g_f^*$.

Water vaporization rate. The rate of water vaporization is obtained from the foregoing results by way of the equation:

$$\dot{m}''_{\mathrm{H_2O}} = \dot{m}'' - \dot{m}''_{\mathrm{C_2H_5OH}}. \qquad (26)$$

The corresponding curve is also plotted in Fig. 8; it shows that, except at very large values of the gas-phase conductance, the water vaporization rate is negative; i.e. steam *condenses*.

Though qualitatively similar to the prediction of the conventional theory (Fig. 6), large quantitative discrepancies exist, particularly when the gas-side conductance is controlling ($g_{\mathrm{I},f}^*$ small).

Discussion. It is evident that the conventional theory totally fails to provide an acceptable prediction of the mass-transfer phenomena for the case in question. However, practical problems often differ from the one studied in two respects, namely: (i) the G- and F-states usually lie on the saturation lines, and (ii) the Lewis number of the liquid phase is usually nearer to zero than to unity.

Before condemning the conventional theory, it is therefore necessary to examine whether its predictions are equally at fault under the conditions just indicated.

4.5. *When the bulk fluid states are saturated*

Relation to the previous problem. Fig. 9 illustrates the new situation that is analysed; the values of f_{G} and of f_{F} are the same as before, 0·495 and 0·675 respectively, but the enthalpies of the G- and F-states are respectively lowered and raised so that their state-points lie on the saturation lines.

Since there has been no change in f_{G} or f_{F}, the predictions of the conventional theory about this new situation are identical with its predictions for the situation of Fig. 7: according to the conventional theory the enthalpies of the G- and F-states are unimportant.

Graphical solution of the problem. The procedure for determining the properties of the transferred substance is the same as before: S and L lie at opposite ends of an isotherm; T lies on GS, produced if necessary, and also on FL; therefore T is found, for a given interface temperature, as the intersection of the corresponding GS and FL lines.

Inspection of Fig. 9 shows that, since G and F

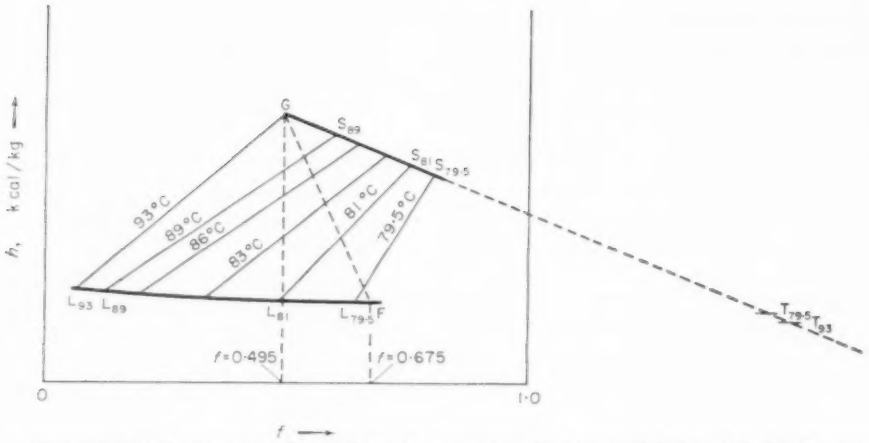

FIG. 9. Illustrating the locations of G and F (on the saturation lines) and of S, L and T, for the example discussed in section 4.5 of the text. Results of calculations are shown in Fig. 10.

now lie on the (approximately straight) saturation lines, the directions of GS and FL can no longer vary much; the consequence is that the locus of T is now extremely short: f_T has the practically constant value of 1·5.

Results. The net rate of mass transfer, and the rates of vaporization of ethanol and water, have been determined in the manner indicated above; they are displayed graphically in Fig. 10. Comparison with Fig. 6 shows a much greater similarity to exist than for the previous case: qualitatively the conventional theory yields good predictions over the whole range of conditions, although there are small quantitative disagreements. The latter spring from two facts: firstly, the conventional theory presumes that $f_T = 1{\cdot}643$, whereas we find $f_T = 1{\cdot}5$ approximately; secondly, the conventional theory neglects the difference between g and g^*.

Discussion. Evidently the conventional theory is not nearly so unreliable when G and F lie on the saturation lines as when they do not. Indeed, if the conventional theory were modified so as to use equation (16a) of the footnote to section 3.1, and $\Delta h_i/\Delta h_j$ were calculated with proper allowance for the non-ideality of the mixtures, its prediction would be quite good.

It should not be forgotten that, in the examples considered rigorously so far, the Lewis numbers have been assumed to equal unity for both the gaseous and the liquid phases. In the next section an example will be worked out for which this condition does not hold.

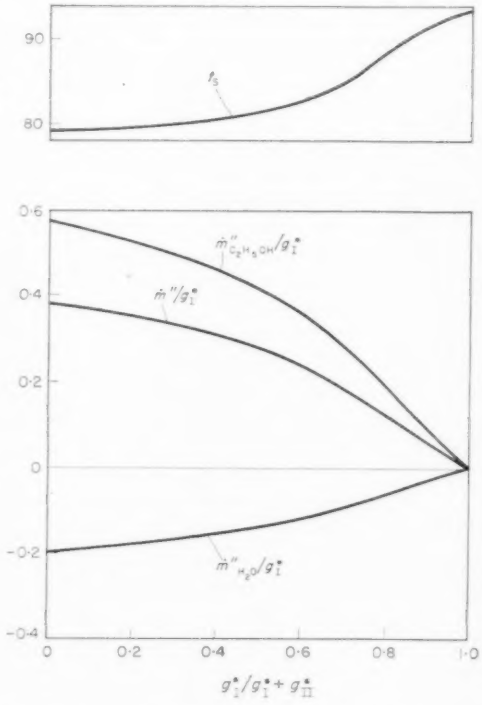

FIG. 10. Results for the rigorous calculation indicated in Fig. 9 (F and G on the saturation line).

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5. RIGOROUS SOLUTION WHEN THE LIQUID PHASE LEWIS NUMBER IS ZERO

5.1. *The problem considered*

The diffusion coefficient of a binary liquid mixture is often one or two orders of magnitude lower than its thermal diffusivity; the Lewis number of the liquid phase may therefore be as low as 0·01. A consequence is that, if the liquid is entirely free of turbulence, the conductance of equation (7) is very much less than that of equation (8).

In order to investigate the influence of this tendency on the nature of the transfer processes, we shall now study an extreme case, that for which the conductance ratio is practically zero in the liquid phase:

$$\frac{g_{\mathrm{II},f}}{g_{\mathrm{II},k}} \ll 1. \tag{27}$$

In other respects, the problem will be similar to that first studied: the G- and F-states will be as illustrated in Figs. 3, 4 and 7; and the gas-phase conductances, $g^*_{\mathrm{I},k}$ and $g^*_{\mathrm{I},f}$, will be taken as equal once more. We shall investigate the influence of the relative values of the gas-side and liquid-side conductances on the surface temperature and on the directions and magnitudes of the mass transfers. Both conventional and rigorous procedures will be used.

5.2. *Predictions of the conventional procedure*

Influence of $g^*_{\mathrm{I},f}/(g^*_{\mathrm{I},f} + g^*_{\mathrm{II},f})$. The fact represented by equation (27) exerts no influence on the conventional theory of mass transfer in a binary mixture; the prediction described in section 3 and displayed in Fig. 6 therefore applies equally well to the present problem.

Influence of the heat-conductance ratio. Now equation (27), taken together with the fact that $g^*_{\mathrm{I},f}$ and $g^*_{\mathrm{I},k}$ are equal, implies that, whenever $g^*_{\mathrm{II},k}$ is of the same order as $g^*_{\mathrm{I},k}$, $g^*_{\mathrm{II},f}$ is very much less than $g^*_{\mathrm{I},f}$. We shall be studying the influence of $g^*_{\mathrm{I},k}/(g^*_{\mathrm{I},k} + g^*_{\mathrm{II},k})$ below; we should therefore remember that, for all values of this quantity greater than zero, the conventional theory predicts that $g^*_{\mathrm{II},f}/g^*_{\mathrm{I},f}$ is zero, and so (Fig. 5) that:

$$f_{\mathrm{S}} = f_{\mathrm{G}}. \tag{28}$$

The intersection of the horizontal line represented in Fig. 5 by equation (28) with the curve representing the equilibrium condition gives the value of f_{L}; it is 0·07. Since f_{T} is assumed to equal 1·643, and f_{F} is equal to 0·675, equations (7) and (22) can be evaluated; thus:

$$\frac{\dot{m}''}{g^*_{\mathrm{II},f}} = -\frac{(0{\cdot}675 - 0{\cdot}07)}{(0{\cdot}07 - 1{\cdot}643)} = 0{\cdot}384 \tag{29}$$

and

$$\frac{\dot{m}''_{\mathrm{C_2H_5OH}}}{g^*_{\mathrm{II},f}} = 0{\cdot}32 \times 1{\cdot}643 = 0{\cdot}627. \tag{30}$$

Consequently, $\dot{m}''/g^*_{\mathrm{II},k}$ and $\dot{m}''_{\mathrm{C_2H_5OH}}/g^*_{\mathrm{II},k}$ are then positive but *close to zero*, by reason of equation (27); it is this result which will be compared with predictions of the rigorous theory, which will now be developed.

5.3. *Rigorous determination of the interface temperature and transferred-substance state*

The transferred-substance composition. Inspection of equations (5) to (8), with particular attention to equation (7), shows that, when $g_{\mathrm{II},f}$ is very much smaller than all the other g's, it is only possible for the transfer rates to be finite if:

$$f_{\mathrm{L}} = f_{\mathrm{T}}. \tag{31}$$

Expressed verbally, this important equation signifies that ethanol and water cross the phase boundary in precisely the same proportion as that which exists between their concentrations in the gas adjacent the interface.

A corollary of equation (31) is that the directions of transfer of both ethanol and water must be the same; for f_{L} is constrained to lie between 0 and 1, so now f_{T} must do likewise.

The location of T. Equation (31) implies that the state-point T on the enthalpy–composition diagram must lie on the vertical line through the point L. This condition, replacing the previous one that F, L and T were co-linear, now enables the location of T to be fixed for any value of the interface temperature, for G, S and T still are co-linear, the Lewis number being unity in the gas phase; T therefore lies on the intersection of GS with the vertical through L. This is shown in Fig. 11.

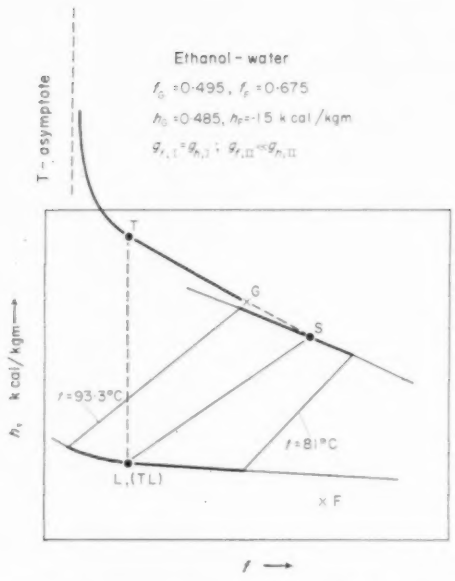

FIG. 11. Illustrating the loci of S and T for the case discussed in section 5 of the text when Lewis number equals to zero for liquid and to unity for gaseous phase. Results of calculations are shown in Fig. 12.

Possible positions of S, L *and* T. The limitations on the values of B, which were mentioned in section 4.2, still prevail. It is therefore easily seen that S, L and T are confined to the loci indicated in Fig. 11 by heavy lines: S cannot lie to the left of G, while L cannot lie to the right of it.

These dispositions ensure that G always lies between S and T so, that $\overline{GS}/\overline{ST}$ is invariably negative. This implies that the direction of net mass transfer is always negative: only condensation occurs.

As has already been seen, f_T lies between 0 and 1. An additional implication of Fig. 11 is therefore that $\dot{m}''_{C_2H_5OH}$ and $\dot{m}''_{H_2O}$ are also invariably negative.

5.4. *Results of the calculation*

Clearly Fig. 11 displays a one-parameter family of situations, for each of which it is possible to determine t_S, $\dot{m}''/g^*_{I,h}$, $\dot{m}''/g^*_{II,h}$, $g^*_{I,h}/(g^*_{I,h} + g^*_{II,h})$, etc. These evaluations have been made; the results are contained in Fig. 12, which has $g^*_{I,h}/(g^*_{I,h} + g^*_{II,h})$ as abscissa.

As already predicted, the curves show that only condensation occurs, the rate increasing as $g^*_{I,h}$ falls relative to $g^*_{II,h}$.

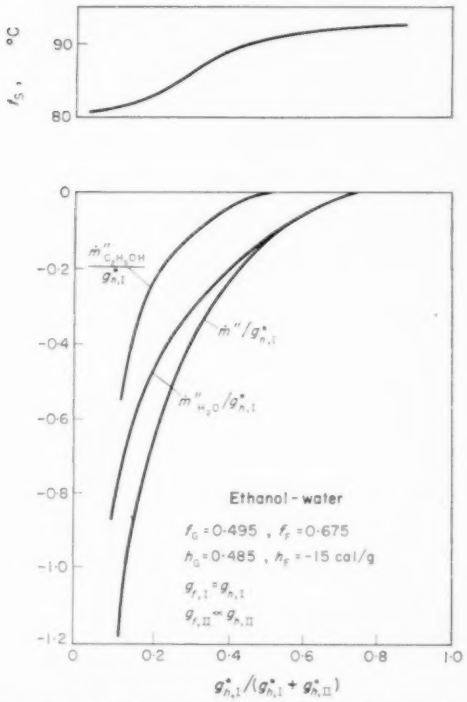

FIG. 12. Results for the example discussed in section 5 of text.

Comparison with the case with a Lewis number of unity for the liquid (*section* 4.1). Fig. 12 is directly comparable with Fig. 8; the only difference in the situations is that $g_{II,f}$ and $g_{II,h}$ are equal for Fig. 8 but obey the inequality (27) for Fig. 12. Any differences between the two sets of curves are therefore purely the consequence of the change of the Lewis number in the liquid.

Inspection of Figs. 8 and 12 shows that the differences are profound. The main one is that the vanishing of $g^*_{II,f}$ suppresses vaporization of ethanol: for it "cannot get out" of the liquid phase. Only at low values of $g^*_{I,h}/(g^*_{I,h} + g^*_{II,h})$

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do the two sets of curves have approximately the same course.†

Comparison with the predictions of the conventional theory. We have already noted that, according to the conventional theory, $\dot{m}''/g^*_{\mathrm{I},h}$, $\dot{m}_{\mathrm{C_2H_5OH}}/g^*_{\mathrm{I},h}$ and $\dot{m}_{\mathrm{H_2O}}/g^*_{\mathrm{I},h}$ are all zero when $g^*_{\mathrm{I},h}/(g^*_{\mathrm{I},h}+g^*_{\mathrm{II},h})$ is finite; consequently the conventional theory predicts that, on a plot such as that of Fig. 8, all three curves should collapse into the base line.

The conventional theory is therefore totally incorrect: it predicts that, because $g^*_{\mathrm{II},f}$ happens to be very small, only a small amount of mass transfer will occur. Yet, with hot vapour in contact with cold liquid, no mere suppression of liquid-phase diffusion can prevent condensation from occurring; indeed, liquid would condense from the vapour onto a cold metal wall, so why should it not also do so onto the surface of the cold F-state liquid? Of course, the condensate will not mix with the bulk of the liquid but will form a layer on its surface, of thickness increasing with time.

5.5. *Case in which the* G- *and* F-*states are saturated*

When the Lewis numbers were unity, it was seen that the conventional theory gave tolerable results when the bulk states of the liquid and vapour were saturated; it is therefore necessary to check whether this is the case for a zero Lewis number in the liquid.

Inspection of Fig. 12 yields the answer, so rendering a detailed calculation unnecessary. We see that the movement of G and F to the saturation lines merely results in a slight change in the shape of the T-locus, which however still has the same asymptote and still terminates at G on the right; all the above qualitative conclusions therefore remain. The conventional theory is still totally wrong.

† There appears to be no physically realistic solution to the equations possible when $g^*_{\mathrm{I},h}/(g^*_{\mathrm{I},h}+g^*_{\mathrm{II},h})$ exceeds the value 0·81. The author is not sure how to interpret this, since it is physically possible to arrange the flow patterns of liquid and gas streams so that this ratio *is* exceeded. This matter requires further study.

6. CONCLUSIONS

(i) Theoretical examination of the heat and mass transfers which occur at the interface separating the two phases of an ethanol–water mixture has shown that the conventional calculation procedure (e.g. [1]) makes tolerably good predictions if the bulk states are saturated *and* the Lewis numbers of liquid and gas are unity.

(ii) If the bulk states are not saturated, even though the Lewis numbers are still unity, the conventional method gives very unreliable predictions; the *directions* of the transfer of the components may be incorrectly predicted.

(iii) When the liquid Lewis number is very small, as often occurs in practice, the conventional theory again makes totally incorrect predictions whether the bulk states are saturated or not.

(iv) Although ten simultaneous equations have to be solved in the rigorous theory, the use of graphical constructions on an enthalpy–composition diagram renders this a relatively simple task. The diagram also provides considerable insight into the processes of mass transfer.

(v) Experimental data which could confirm or deny the above conclusions do not appear to be available; the process in question is of sufficiently central importance in chemical engineering to deserve further study.

(vi) The standard formulation for mass-transfer problems [4] appears to be flexible enough to accommodate all the phenomena of binary-mixture mass transfer, provided that subscripts are introduced to distinguish the two phases. In particular, the concept of the "transferred substance" (T-state) is a helpful one.

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REFERENCES

1. J. H. Perry (Editor) *Chemical Engineers' Handbook*. Sections 8 and 9, McGraw-Hill, New York (1950).
2. T. K. Sherwood and R. L. Pigford *Absorption and Extraction* (2nd Ed.). McGraw-Hill, New York (1952).

3. R. E. TREYBAL, *Mass transfer operations*, McGraw-Hill, New York (1955).
4. D. B. SPALDING, A standard formulation of the steady convective mass-transfer problem. *Int. J. Heat Mass Transfer*, **1**, 192 (1960).
5. F. BOŠNJAKOVIĆ, *Technische Thermodynamik*, Vol. I. Steinkopf, Leipzig (1935).
6. A. P. COLBURN and T. B. DREW, The condensation of mixed vapours. *J. Amer. Inst. Chem. Engrs*, **33**, 197–212 (1937).

Résumé—Cet article présente une analyse théorique rigoureuse du transport de chaleur et de masse qui intervient sur un élément de la surface de séparation des phases liquide et gazeuse d'un mélange binaire. Des solutions des équations sont présentées pour des mélanges eau–alcool éthylique dans les troise cas suivants: (1) Les milieux gazeux et liquide ne sont pas saturés et les deux nombres de Lewis sont égaux à l'unité; (2) Les phases sont saturés et les deux nombres de Lewis sont égaux à l'unité; (3) Le nombre de Lewis du liquide est voisin de zéro tandis que celui du gaz ést égal à l'unité, les milieux étant saturés ou non. Les résultats sont comparés aux prévisions des solutions approchées habituellement utilisées (transport équimoléculaire).

La théorie habituelle ne donne des résultats dont la précision est acceptable que dans le cas 2; dans les autres cas ses solutions n'ont presque pas de rapport avec celles de la théorie rigoureuse.

La théorie rigoureuse montre que, dans les exemples étudiés, il peut se produire une condensation ou une vaporisation très nette suivant les valeurs de la conductance et les états des milieux. Dans quelques cas, le composant le plus volatil se condense bien qu'il ait une concentration en phase gazeuse plus élevée à la surface de séparation que dans le milieu continu gazeux.

Zusammenfassung—Für den Wärme- und Stoffübergang an einem Element der Trennschicht zwischen Flüssigkeits- und Gasphase eines binären Gemisches wird eine strenge theoretische Analyse angegeben. Die Lösungen der Gleichungen erstrecken sich auf Äthanol–Wasser Gemische für drei verschiedene Fälle, nämlich—(i) die Flüssigkeits und Gasströme sind nicht gesättigt und beide Lewiszahlen haben die Grösse eins; (ii) Flüssigkeits- und Gasströme sind gesättigt, beide Lewiszahlen sind gleich eins; (iii) die Lewiszahl der Flüssigkeit ist nahezu Null, während die des Gases gleich eins ist und die Ströme entweder gesättigt sind oder nicht. Die Ergebnisse werden mit jenen verglichen, die aus gewöhnlich verwendeten Näherungsbetrachtungen (äquimolare Übertragung) gewonnen werden (z.B. [1]). Die konventielle Theorie ergibt nur im Falle (ii) leidliche Genauigkeit; in den anderen Fällen stehen ihre Ergebnisse in nahezu keinem Zusammenhang mit denen der strengen Theorie. Nach der strengen Theorie kann in den betrachteten Beispielen—abhängig von den Leitfähigkeitswerten und den Zuständen im Innern der Flüssigkeit bzw. des Gases—entweder reine Kondensation oder reine Verdampfung auftreten. In einigen Fällen kondensiert die flüchtigere Komponente, obwohl an der Trennfläche eine höhere Gasphasenkonzentration vorliegt als im Innern des Gases.

Аннотация—Предложена совместная система уравнений, описывающая установившийся тепло-и массоперенос через промежуточный элемент, отделяющий жидкую фазу от газообразной, состоящей из паров той же жидкости. Описан графический метод решения предложенной системы уравнений. Приведены численные примеры расчётов по предложенному методу. Полученные результаты сопоставляются с общепринятым методом расчёта.

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ON THE THERMODYNAMICS OF IRREVERSIBLE HEAT AND MASS TRANSFER

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Abstract—"Non-equilibrium thermodynamics" can serve as a basis for a macroscopic treatment of the irreversible phenomena. It is shown for a system in which heat conduction, diffusion, viscous flow, etc. can take place, how one can find a complete set of differential equations, describing the irreversible behaviour, from the relevant conservation laws, entropy balance and phenomenological equations.

1. INTRODUCTION

THE entropy of a system can change in two ways: either by external supply of heat or matter, or by production of entropy inside the system. The first change may be positive, zero or negative, depending on the interaction of the system with its surroundings. The second change, the entropy production, which arises from irreversible processes taking place inside the system, is positive according to the second law of thermodynamics. In classical thermodynamics one is usually satisfied with this statement in the form of an inequality. In the so-called "thermodynamics of irreversible processes" or "non-equilibrium thermodynamics" however, one wishes to derive an expression for the magnitude of the entropy production which characterizes the irreversibility of the phenomena.

The procedure followed in non-equilibrium thermodynamics consists of first establishing the various conservation laws of macroscopic physics: the laws of conservation of matter, momentum, angular momentum and energy. (In section 2 of this paper we give the forms which these laws take for an isotropic fluid in which heat conduction, diffusion of matter and viscous flow can occur. In sections 4 and 5 we study the effects caused by chemical reactions, relaxation processes and external forces.) Then Gibbs' entropy law is written and an entropy balance equation is established with the help of the conservation laws. This balance equation contains as a source term the entropy production strength. The expression for the entropy production found in this way consists of a sum of contributions arising from heat conduction, diffusion, viscous flow and chemical reactions (sections 3, 4 and 5). Each of these contributions consists of a product of a flux (such as the heat flow or the diffusion flow) and a "thermodynamic force" (such as the temperature gradient or the concentration gradient). One can establish linear relationships (called phenomenological equations) between these fluxes and thermodynamic forces (section 6). The coefficients occurring in these equations are heat conductivities, diffusion coefficients and the like. Certain relations exist between these coefficients as a result of time reversal invariance (the Onsager relations) and of possible spatial symmetries of the system (the Curie principle). Finally one obtains a complete set of differential equations describing the system by inserting the phenomenological equations into the conservation and entropy laws and by writing down the appropriate equations of state (section 7).

A variety of irreversible phenomena have been studied by the method outlined above [1–3], e.g., besides the processes mentioned already, thermal diffusion, thermo-electricity, heat conduction in reacting mixtures, electric conduction, electric and magnetic relaxation, sound dispersion and absorption, etc.

Two points of special interest must still be mentioned. In the first place the use of thermodynamical concepts, such as entropy, outside equilibrium needs justification and the Onsager

relations must be proved. Both these problems can be discussed on the basis of statistical mechanics, e.g., the theory of stochastic processes or the kinetic theory of matter [3].

The second point of interest is the practical application of the theory. To this end the differential equations referred to above must be solved by using the necessary initial and boundary conditions. Progress has been made in obtaining analytical and numerical results of technical importance these last years [4].

2. THE CONSERVATION LAWS

We shall establish the basic conservation laws of mass, momentum, angular momentum and energy in a way which is appropriate for non-uniform systems: in the so-called "local" form, i.e. with physical quantities as field variables which are continuous functions of the space co-ordinates and of time.

(a) *Conservation of mass* is expressed by

$$\frac{\partial \rho_k}{\partial t} = -\text{div } \rho_k \mathbf{v}_k, \quad (k = 1, 2, \ldots, n), \quad (1)$$

where ρ_k is the density of component k of a mixture of n non-reacting components, and $\mathbf{v}_k$ its velocity (both continuous functions of space variables and of time). An alternative form of (1) is

$$\rho \frac{\mathrm{d}c_k}{\mathrm{d}t} = -\text{div } \mathbf{J}_k, \quad (k = 1, 2, \ldots, n), \quad (2)$$

where $c_k = \rho_k/\rho$ is the mass fraction of component k, with ρ the total density $\Sigma_k \rho_k$ and where

$$\mathbf{J}_k = \rho_k(\mathbf{v}_k - \mathbf{v}), \quad (k = 1, 2, \ldots, n) \quad (3)$$

is the diffusion flow of k, counted with respect to the centre of gravity velocity $\mathbf{v}$, which is defined by $\rho\mathbf{v} = \Sigma_k \rho_k \mathbf{v}_k$. The time derivative in (2) is the "barycentric" substantial derivative $\partial/\partial t + \mathbf{v} \cdot \text{grad}$.

(b) *Conservation of momentum* is given by the equation

$$\frac{\partial \rho\mathbf{v}}{\partial t} = -\text{Div } (\rho\mathbf{v}\mathbf{v} + \boldsymbol{P}), \quad (4)$$

where $\rho\mathbf{v}$ is the momentum density. The momentum flux at the right-hand side contains the dyad $\mathbf{v}\mathbf{v}$ (a tensor with α, β components $v_\alpha v_\beta$ where α and β are x, y or z) and the pressure tensor $\boldsymbol{P}$. Equation (4) is mathematically equivalent with the equation of motion in a continuous medium

$$\rho \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\text{Div } \boldsymbol{P}. \quad (5)$$

We have supposed here that no external forces act on the system (cf. however section 5).

One finds a balance equation for the kinetic energy by scalar multiplication of (5) by $\mathbf{v}$

$$\rho \frac{\mathrm{d}\frac{1}{2}\mathbf{v}^2}{\mathrm{d}t} = -\mathbf{v} \cdot \text{Div } \boldsymbol{P} = -\text{div } \boldsymbol{P} \cdot \mathbf{v} + \tilde{\boldsymbol{P}} : \text{Grad } \mathbf{v}, \quad (6)$$

where $\tilde{\boldsymbol{P}}$ is the transposed matrix of $\boldsymbol{P}(\tilde{P}_{\alpha\beta} = P_{\beta\alpha})$.

Similarly a balance equation for (external) angular momentum per unit mass $\mathbf{L} \equiv \mathbf{x} \wedge \mathbf{v}$ follows by vectorial multiplication of (5) by $\mathbf{v}$:

$$\rho \frac{\mathrm{d}L_{\alpha\beta}}{\mathrm{d}t} = -\sum_\gamma \frac{\partial}{\partial x_\gamma}(x_\alpha P_{\gamma\beta} - x_\beta P_{\gamma\alpha}) + P_{\alpha\beta} - P_{\beta\alpha}, (\alpha, \beta = x, y, z). \quad (7)$$

For convenience we have here represented the axial vector $\mathbf{L}$ by an antisymmetric tensor according to the usual convention $L_x \equiv L_{yz} = -L_{zy}$ (cycl.). The summation over γ extends over x, y and z.

(c) *Conservation of angular momentum* is expressed by the equation

$$\rho \frac{\mathrm{d}J_{\alpha\beta}}{\mathrm{d}t} = -\sum_\gamma \frac{\partial}{\partial x_\gamma}(x_\alpha P_{\gamma\beta} - x_\beta P_{\gamma\alpha}), \quad (8)$$

where $\mathbf{J}$ (again representing this axial vector by an antisymmetric tensor) is the angular momentum per unit mass. We split it into an external angular momentum $\mathbf{L}$ and an internal angular momentum $\mathbf{S}$. The balance equation for the latter is obtained by subtraction of (7) from (8), which yields

$$\rho \frac{\mathrm{d}\mathbf{S}}{\mathrm{d}t} = -2\mathbf{P}^a, \quad (9)$$

where $\mathbf{P}^a$ is the antisymmetric part $\frac{1}{2}(\boldsymbol{P} - \tilde{\boldsymbol{P}})$ of

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the pressure tensor $\boldsymbol{P}$, written as an axial vector. One can write $\mathbf{S}$ as $\Theta\,\boldsymbol{\omega}$, with Θ the moment of inertia per unit mass and $\boldsymbol{\omega}$ the angular velocity. Then the local rotational energy per unit mass is $u_r = \frac{1}{2}\Theta\boldsymbol{\omega}^2$, for which one finds a balance equation from (9):

$$\rho \frac{\mathrm{d}u_r}{\mathrm{d}t} = -2\boldsymbol{\omega} \cdot \mathbf{P}^a. \tag{10}$$

(d) *Conservation of energy* is expressed by

$$\rho \frac{\mathrm{d}e}{\mathrm{d}t} = -\mathrm{div}\,(\boldsymbol{P} \cdot \mathbf{v} + \mathbf{J}_q), \tag{11}$$

where e is the total energy per unit mass and $\mathbf{J}_q$ the heat flow. If we define the internal energy u by splitting e in the following way

$$e = \tfrac{1}{2}\mathbf{v}^2 + u_r + u, \tag{12}$$

we can obtain the balance equation for u by subtracting (6) and (10) from (11):

$$\rho \frac{\mathrm{d}u}{\mathrm{d}t} = -\widetilde{\boldsymbol{P}} : \mathrm{Grad}\,\mathbf{v} + 2\mathbf{P}^a \cdot \boldsymbol{\omega} - \mathrm{div}\,\mathbf{J}_q. \tag{13}$$

Let us now split the pressure tensor in the following way

$$\boldsymbol{P} = p\boldsymbol{U} + \boldsymbol{\Pi}, \tag{14}$$

where p is the equilibrium pressure ($\boldsymbol{U}$ is the unit matrix), and where $\boldsymbol{\Pi}$ is called the viscous pressure tensor. We can split the latter into three parts

$$\boldsymbol{\Pi} = \Pi\boldsymbol{U} + \mathring{\boldsymbol{\Pi}}^s + \boldsymbol{\Pi}^a, \tag{15}$$

where Π is one third of the trace, $\mathring{\boldsymbol{\Pi}}^s$ is the symmetric part with zero trace and $\boldsymbol{\Pi}^a$ the antisymmetric part. The latter, which according to (14) and (15) is equal to $\boldsymbol{P}^a$, can again be represented by an axial vector $\mathbf{P}^a$. Similarly we can split the velocity gradient tensor

$$\mathrm{Grad}\,\mathbf{v} = \tfrac{1}{3}\,(\mathrm{div}\,\mathbf{v})\,\boldsymbol{U} + (\mathring{\mathrm{G}}\mathrm{rad}\,\mathbf{v})^s + (\mathrm{Grad}\,\mathbf{v})^a, \tag{16}$$

where the antisymmetric part corresponds to the axial vector $\frac{1}{2}\,\mathrm{rot}\,\mathbf{v}$. The balance equation for the internal energy (13) can now be written as

$$\rho\left(\frac{\mathrm{d}u}{\mathrm{d}t} + p\frac{\mathrm{d}v}{\mathrm{d}t}\right) = -\Pi\,\mathrm{div}\,\mathbf{v} - \mathring{\boldsymbol{\Pi}}^s : (\mathring{\mathrm{G}}\mathrm{rad}\,\mathbf{v})^s - \boldsymbol{\Pi}^a \cdot (\mathrm{rot}\,\mathbf{v} - 2\boldsymbol{\omega}) - \mathrm{div}\,\mathbf{J}_q. \tag{17}$$

Here $v \equiv \rho^{-1}$ is the specific volume, and use has been made of the fact that, according to (1), $\mathrm{d}\mathbf{v}/\mathrm{d}t$ is equal to $\rho^{-1}\,\mathrm{div}\,\mathbf{v}$. This result represents the first law of thermodynamics for our system.

3. ENTROPY LAW AND ENTROPY BALANCE

For the system considered we can express the entropy per unit mass s in terms of the local internal energy, specific volume and mass fractions: $s = s(u, v, c_k)$. We shall assume that locally s is the same function of these variables as it is in equilibrium i.e. more specifically we assume that Gibbs' entropy law is valid along the line of motion of the centre of gravity of a mass element:

$$T\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\mathrm{d}u}{\mathrm{d}t} + p\frac{\mathrm{d}v}{\mathrm{d}t} - \Sigma_k \mu_k \frac{\mathrm{d}c_k}{\mathrm{d}t}, \tag{18}$$

where μ_k is called the chemical potential of component k. If we substitute the mass law (2) and the energy law (17) into this equation we obtain a balance equation for the entropy, which has the form

$$\rho\frac{\mathrm{d}s}{\mathrm{d}t} = -\mathrm{div}\,\mathbf{J}_s + \sigma, \tag{19}$$

with an "entropy flow"

$$\mathbf{J}_s = (\mathbf{J}_q - \Sigma_k \mu_k \mathbf{J}_k)/T \tag{20}$$

and an "entropy production"

$$\left.\begin{aligned}\sigma = &-\mathbf{J}_q \cdot \frac{\mathrm{grad}\,T}{T^2} - \sum_k \mathbf{J}_k \cdot \mathrm{grad}\left(\frac{\mu_k}{T}\right) \\ &- \frac{\Pi\,\mathrm{div}\,\mathbf{v}}{T} - \frac{\mathring{\boldsymbol{\Pi}}^s : (\mathring{\mathrm{G}}\mathrm{rad}\,\mathbf{v})^s}{T} \\ &- \frac{\boldsymbol{\Pi}^a \cdot (\mathrm{rot}\,\mathbf{v} - 2\boldsymbol{\omega})}{T} \geqslant 0.\end{aligned}\right\} \tag{21}$$

According to (19) the entropy can change in two ways: in the first place by external *supply of heat or matter* as expressed by the first term on the right-hand side, which contains the heat flow $\mathbf{J}_q$ and the diffusion flow $\mathbf{J}_k$, as (20) shows. In the second place the entropy can change by *internal production* σ. This entropy production, which is the second term on the right-hand side of (19), is positive (or zero) according to the second law of thermodynamics. It is a measure

for the irreversibility of the processes which take place inside the system. (In particular it vanishes at thermodynamic equilibrium.) As its expression (21) shows, it contains five contributions of which the first arises from heat conduction, the second from diffusion of matter and the other three from viscous flow. Each term is the product of a flux (heat flow $\mathbf{J}_q$, diffusion flow $\mathbf{J}_k$, viscous pressure tensor components) and a so-called "thermodynamic force" (temperature gradient, gradient of chemical potential, velocity gradients). It may be noted here already that the first two fluxes and thermodynamic forces are (polar) vectors, the third term contains scalars, the fourth symmetric tensors with zero trace and the fifth axial vectors. We may also remark that we shall see in section 6 that the last three terms of (21) are connected to volume viscosity, shear viscosity and rotational viscosity respectively.

4. CHEMICAL REACTIONS AND RELAXATION PHENOMENA

If a chemical reaction can occur, we must slightly modify the preceding formalism. In the mass laws (1) and (2) we must add a source term at the right-hand side, which accounts for the chemical production of component k in mass per unit volume and unit time. We can write this term as $\nu_k J_c$ where ν_k divided by the molecular mass M_k of k is the stoichiometric coefficient of k in the chemical reaction, and where J_c is called the chemical reaction rate. The latter quantity can alternatively be written as $\rho \mathrm{d}\xi_c/\mathrm{d}t$, where ξ_c is called the chemical "progress variable". In this way (2) becomes

$$\rho \frac{\mathrm{d}c_k}{\mathrm{d}t} = -\operatorname{div} \mathbf{J}_k + \nu_k \rho \frac{\mathrm{d}\xi_c}{\mathrm{d}t}, \quad (k = 1, 2, \ldots, n). \tag{22}$$

Summation over k yields the same equation as before, because, since the total mass is conserved in a chemical reaction, one has $\Sigma_k \nu_k = 0$.

The entropy balance follows now from (18) with (17) and (22). One obtains (19) with (20) for the entropy flux, but in the entropy production (21) an additional term appears:

$$\sigma_c = -\frac{\rho}{T} \frac{\mathrm{d}\xi_c}{\mathrm{d}t} \Sigma_k \mu_k \nu_k = -\frac{\rho}{T} \frac{\mathrm{d}\xi_c}{\mathrm{d}t} A_c, \tag{23}$$

where A_c, defined as $\Sigma_k \mu_k \nu_k$, is called the chemical affinity. The generalization to the case of more than one single chemical reaction is straightforward: one simply obtains sums over chemical reactions in the last term of (22) and in (23).

The occurrence of relaxation processes in a system gives rise to terms in the entropy production similar to those arising from chemical reactions. If a relaxation phenomenon takes place, this means that some "internal state variable" ξ is needed to describe the system. Then the entropy s depends not only on u, v and c_k but also on the internal variable ξ. (The generalization to several internal variables, i.e. to several relaxation processes, is again straightforward.) The Gibbs' entropy law (18) becomes in this way

$$T \frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\mathrm{d}u}{\mathrm{d}t} + p \frac{\mathrm{d}v}{\mathrm{d}t} - \Sigma_k \mu_k \frac{\mathrm{d}c_k}{\mathrm{d}t} - A \frac{\mathrm{d}\xi}{\mathrm{d}t}, \tag{24}$$

where A is called relaxation affinity. The entropy production contains therefore an additional term, arising from relaxation,

$$\sigma_r = -\frac{\rho}{T} \frac{\mathrm{d}\xi}{\mathrm{d}t} A, \tag{25}$$

completely analogous to (23). That this must be so is clear because we can consider a chemical reaction as a special case of a relaxation phenomenon.

5. THE CASE OF EXTERNAL FORCES

In the presence of external forces, a source term has to be added in the right-hand side of the equation of motion (5) of the form $\Sigma_k \rho_k \mathbf{F}_k$, where $\mathbf{F}_k$ is the force per unit mass acting on substance k. This means that the balance of kinetic energy is modified, since it is derived from the equation of motion. In the case of *conservative forces* defined as

$$\mathbf{F}_k = -\operatorname{grad} \psi_k, \quad (\text{with } \partial\psi_k/\partial t = 0), \tag{26}$$

one can easily derive a balance equation for the potential energy per unit mass $\psi = \Sigma_k c_k \psi_k$. The internal energy u is now defined by subtracting not only the kinetic energies $\frac{1}{2}\mathbf{v}^2$ and u_r from the total energy e, but also the potential energy ψ. Then one obtains, proceeding in the same way

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as above, an extra term in the entropy production of the form

$$\sigma_F = \frac{1}{T}\Sigma_k \mathbf{J}_k \cdot \mathbf{F}_k, \tag{27}$$

which can be combined with the diffusion term because it also contains the diffusion flows $\mathbf{J}_k$.

Let us write down also the results for a system in the presence of *non-conservative forces*. The most important example of such forces is that of electromagnetic forces, for which one has the Lorentz expression

$$\mathbf{F}_k = z_k (\mathbf{E} + c^{-1}\, \mathbf{v}_k \wedge \mathbf{B}), \tag{28}$$

where z_k is the electric charge per unit mass of component k, and where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields. The procedure, outlined above (one must now also establish balances of electromagnetic momentum and energy by means of Maxwell's equations), leads to a new term in the entropy production

$$\sigma_L = \frac{1}{T}\mathbf{i} \cdot \left(\mathbf{E} + \frac{\mathbf{v}}{c} \wedge \mathbf{B}\right), \tag{29}$$

where $\mathbf{i} = \Sigma_k z_k \mathbf{J}_k$ is the conduction current of electricity. In the preceding we have supposed that the material is not polarizable, i.e. we have supposed that the electric and magnetic polarizations per unit mass $\mathbf{p}$ and $\mathbf{m}$ vanish. If this is not the case one has the additional state variables $\mathbf{p}$ and $\mathbf{m}$, and the form of the entropy law becomes

$$T\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\mathrm{d}u}{\mathrm{d}t} + p\frac{\mathrm{d}v}{\mathrm{d}t} - \mathbf{E}\cdot\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} - \mathbf{B}\cdot\frac{\mathrm{d}\mathbf{m}}{\mathrm{d}t} - \Sigma_k \mu_k \frac{\mathrm{d}c_k}{\mathrm{d}t}. \tag{30}$$

The calculation now leads finally to two more terms in the entropy production:

$$\sigma_p = \frac{\rho}{T}\left(\Delta\mathbf{E}' \cdot \frac{\mathrm{d}\mathbf{p}'}{\mathrm{d}t} + \Delta\mathbf{B}' \cdot \frac{\mathrm{d}\mathbf{m}'}{\mathrm{d}t}\right), \tag{31}$$

where primed quantities are the fields and polarizations measured by an observer with velocity $\mathbf{v}$. The quantities $\Delta\mathbf{E}'$ and $\Delta\mathbf{B}'$ are the deviations of the values of the actual fields from their equilibrium values, which would correspond to the polarizations $\mathbf{p}$ and $\mathbf{m}$. The two terms of (31) are related to the irreversible phenomena of electric and magnetic relaxation.

6. THE PHENOMENOLOGICAL EQUATIONS, THE CURIE PRINCIPLE AND THE ONSAGER RELATIONS

At thermodynamic equilibrium the thermodynamic forces are zero according to the well-known rules of thermodynamics. In conformity with the concept of equilibrium we assume that simultaneously also the fluxes vanish. It is then plausible to make the assumption that, in the first approximation, outside equilibrium the components of the fluxes are linearly related to the components of the thermodynamic forces. Such relations, which are called the phenomenological equations, have been empirically verified for a large class of irreversible processes.

According to their transformation properties under rotations and inversions, we subdivided the different fluxes and thermodynamic forces into scalars, polar vectors, axial vectors and symmetric tensors with zero trace. Now the symmetry properties of the system influence the linear laws in the sense that not all fluxes couple with all thermodynamic forces. This statement, which is known as the *Curie principle*, can be proved for any type of symmetry by means of the theory of invariants [3]. For isotropic systems the result is that a certain force cannot give rise to a flow of different tensorial character. We shall write down the phenomenological equations for the fluxes and thermodynamic forces of (21) which we first rewrite, eliminating $\mathbf{J}_n$ by means of the relation $\Sigma_{k=1}^{n} \mathbf{J}_k = 0$ which results from (3), in the following fashion

$$\sigma = -\mathbf{J}_q \cdot \frac{\operatorname{grad} T}{T} - \sum_{k=1}^{n-1} J_k \cdot \operatorname{grad}\frac{\mu_k - \mu_n}{T} - \frac{\Pi \operatorname{div}\mathbf{v}}{T} - \frac{\overset{\circ}{L}{}^{s} : (\operatorname{Grad}\mathbf{v})^s}{T} - \frac{\Pi^a \cdot (\operatorname{rot}\mathbf{v} - 2\omega)}{T}. \tag{32}$$

Then, according to what was said above, we have the phenomenological equations:

$$\mathbf{J}_q = -L_{qq}\frac{\operatorname{grad} T}{T^2} - \sum_{k=1}^{n-1} L_{qk} \operatorname{grad}\frac{\mu_k - \mu_n}{T}, \tag{33}$$

$$\mathbf{J}_i = -L_{iq}\frac{\operatorname{grad} T}{T^2} - \sum_{k=1}^{n-1} L_{ik}\operatorname{grad}\frac{\mu_k - \mu_n}{T}, \quad (i = 1, 2, \ldots, n-1), \tag{34}$$

$$\Pi = -L_v\frac{\operatorname{div}\mathbf{v}}{T} \equiv -\eta_v \operatorname{div}\mathbf{v}, \tag{35}$$

$$\mathring{\Pi}^s = -L\frac{(\text{Grâd}\,\mathbf{v})}{T} \equiv -2\eta\,(\text{Grâd}\,\mathbf{v})^s, \tag{36}$$

$$\Pi^a = -L_r\frac{\operatorname{rot}\mathbf{v} - 2\omega}{T} \equiv -\eta_r(\operatorname{rot}\mathbf{v} - 2\omega). \tag{37}$$

Equations (33) and (34) contain vectorial fluxes and forces, (35) scalars, (36) symmetric tensors with zero trace, and (37) axial vectors.

All coefficients have a particular physical meaning. Thus L_{qq}/T^2 is the heat conductivity, the L_{iq} are related to the thermal diffusion coefficients, the L_{qi} are related to the Dufour coefficients, the L_{ik} are related to the diffusion coefficients, η_v is the volume viscosity, η is the shear viscosity and η_r is the rotational viscosity. With the help of some transformations which leave the entropy production invariant one can introduce diffusion flows which are counted with respect to other reference velocities as the centre of gravity velocity $\mathbf{v}$, employed here. The gradients of the chemical potentials can alternatively be expressed in terms of various composition parameters such as the mass fractions c_k, the molar fractions, partial densities and the like. It is possible to develop in this way a completely rigorous theory of diffusion coefficients for any number n of chemical components [3].

If a relaxation phenomenon (or a chemical reaction) occurs, we have the additional term (25) in the entropy production. Since it is a product of scalars, we must consider it together with the volume viscosity term; this yields the phenomenological equations:

$$\Pi = -L_v\frac{\operatorname{div}\mathbf{v}}{T} - L_{vr}\frac{\rho}{T}A, \tag{38}$$

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} = -L_{rv}\frac{\operatorname{div}\mathbf{v}}{T} - L_r\frac{\rho}{T}A. \tag{39}$$

We have again the volume viscosity $\eta_v = L_v/T$ and furthermore a relaxation coefficient L_r, which can be connected to a characteristic time because we can write for (39), omitting for the moment the term with div $\mathbf{v}$,

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} = -L_r\frac{\rho}{T}A = -L_r\frac{\rho}{T}\frac{\partial A}{\partial\xi}(\xi - \xi_0), \tag{40}$$

where ξ_0 is the equilibrium value of ξ. This equation shows that

$$\tau \equiv \frac{T}{L_r\rho}\frac{\partial\xi}{\partial A} \tag{41}$$

is a characteristic time, called the relaxation time. Finally we have in (38) and (39) two cross-coefficients L_{rv} and L_{vr}.

If external forces act on the system, we have according to (27) a term $\mathbf{F}_k/T$ besides the original thermodynamic force $-\operatorname{grad}(\mu_k/T)$. If, for example, $\mathbf{F}_k$ is the centrifugal force, then we can derive the theory of sedimentation in rotating systems. If $\mathbf{F}_k$ is of electromagnetic origin, then the theory of electric conduction, of galvanomagnetic and thermomagnetic phenomena and of electric and magnetic relaxation can be developed.

Another simplification of the scheme of coefficients can be obtained with the help of the *Onsager reciprocal relations*, which play a central role in non-equilibrium thermodynamics. These relations can be derived with the help of statistical mechanical notions from the property of "time reversal invariance" (or "microscopic reversibility") of the microscopic mechanical laws (i.e. invariance for $t \longleftrightarrow -t$). In the cases considered here the Onsager relations read then

$$L_{iq} = L_{qi} \qquad (i = 1, 2, \ldots, n-1), \tag{42}$$

$$L_{ik} = L_{ki}, \qquad (i, k = 1, 2, \ldots, n-1), \tag{43}$$

$$L_{vr} = -L_{rv}. \tag{44}$$

Relation (42) connects the thermal diffusion coefficient and the Dufour coefficient. Relation (43) is a connexion between diffusion coefficients. For binary mixtures ($n = 2$) there is only one coefficient L_{11} and thus no Onsager relation; for ternary mixtures ($n = 3$) one has four coefficients L_{11}, L_{12}, L_{21} and L_{22}, with one Onsager relation $L_{12} = L_{21}$ of the type (43). The minus sign in (44) is a consequence of the fact that, whereas A is a so-called even variable (its sign

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does not change when the individual particle velocities are reversed), the force div $\mathbf{v}$ has odd character with respect to particle velocity reversal.

(We shall not consider here the case of external magnetic fields, where the form of the Onsager relations is slightly modified.)

7. THE DIFFERENTIAL EQUATIONS

If the phenomenological equations (33)–(37) are substituted into the $n - 1$ conservation laws for matter (2) and $\partial\rho/\partial t = -\operatorname{div}\rho\mathbf{v}$ (which follows from (1) by summation over k), the equation of motion (5), the equation of internal angular momentum (9) and the equation of internal energy (17), one obtains a set of $n + 7$ partial differential equations for the $n + 7$ independent variables: the mass fractions $c_1, c_2, \ldots, c_{n-1}$, the density ρ, the cartesian components of $\mathbf{v}$ and ω, and the temperature T. The equations of state allow one to express the equilibrium pressure p, the energy u and the chemical potentials μ_k, occurring in the equations, in terms of those independent variables. The set of equations, established in this way, describes completely the time behaviour of the system for specified initial and boundary conditions.

For a one-component isotropic fluid the differential equations become, in the way outlined above,

$$\frac{\partial\rho}{\partial t} = -\operatorname{div}\rho\mathbf{v}, \tag{45}$$

$$\rho\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\operatorname{grad} p + \eta\Delta\mathbf{v} + (\tfrac{1}{3}\eta + \eta_v)\operatorname{grad}\operatorname{div}\mathbf{v} + \eta_r\operatorname{rot}(2\omega - \operatorname{rot}\mathbf{v}), \tag{46}$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = -\frac{4\eta_r}{\rho\Theta}(\omega - \tfrac{1}{2}\operatorname{rot}\mathbf{v}), \tag{47}$$

$$\rho\left(\frac{\mathrm{d}u}{\mathrm{d}t} + p\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\right) = \eta_v(\operatorname{div}\mathbf{v})^2 + 2\eta\,(\operatorname{Grad}\mathbf{v})^s : (\operatorname{Grad}\mathbf{v})^s + \eta_r(\operatorname{rot}\mathbf{v} - 2\omega)^2 + \lambda\Delta T. \tag{48}$$

We have supposed here that the phenomenological coefficients, the three viscosities η_v, η and η_r, and the heat conductivity λ, are constants. These equations, which are the conservation of mass law (45), the equation of motion (46), the internal angular momentum law (47) and the internal energy balance (48), must be supplemented by the equations of state

$$p = p(\rho, T), \tag{49}$$

$$u = u(\rho, T). \tag{50}$$

The set of equations (45)–(50) describes the general "thermohydrodynamic" behaviour of an isotropic fluid. It contains as a special case ordinary hydrodynamics which is based on (45)–(48) alone by assuming that either isothermal or adiabatic conditions are fulfilled. In both cases p is a function of ρ only, so that the hydrodynamic behaviour is already given by (45)–(47), if $p = p(\rho)$ is substituted. We note that (46) is the well-known Navier–Stokes equation, supplemented by a rotational term, and that the first two terms of the right-hand side of (48) are the Rayleigh dissipation function. The complete set of equations contains also the theory of heat conduction. In particular equation (48) becomes Fourier's differential equation for a system at rest:

$$\rho\frac{\partial u}{\partial t} = \lambda\Delta T, \tag{51}$$

where $\partial u/\partial t = c_v\partial T/\partial t$ with c_v the specific heat per unit mass.

If a chemical reaction occurs we need the phenomenological equations (38) and (39), and the mass law (22) instead of (2).

If a relaxation process takes place we need also the phenomenological equations (38) and (39), but as a mass law equation (2). A new equation of state is required besides the others, mentioned above, which gives the affinity A in terms of the independent state variables, amongst which ξ now occurs. One can also develop the theory of sound dispersion and absorption, caused by a relaxation process and by heat conduction and viscous flow, with the help of the complete set of equations. The important result can then be derived that for sound frequencies ν such that

$\nu\tau \ll 1$, where τ is the relaxation time (41), the relaxation phenomenon can formally be described as an effective volume viscosity.

As indicated in the beginning of this section, for multi-component systems, where diffusion occurs, the set of differential equations becomes more complicated. It may be said that non-equilibrium thermodynamics permits one to study various irreversible processes as heat conduction, diffusion of matter and viscous flow from a single point of view. It encompasses a number of phenomenological theories such as the hydrodynamics of viscous fluids, the theory of diffusion and the theory of heat conduction.

REFERENCES

1. I. PRIGOGINE, *Etude thermodynamique des phénomènes irréversibles*, Desoer, Liège (1947).
2. S. R. DE GROOT, *Thermodynamics of irreversible processes*, North-Holland Publishing Company, Amsterdam, New York (1951);
   S. R. DE GROOT, *Termodinamika neobratimykh protsessov*, Gosudarstbennoe Izdatel'stbo Tehniko-teoretičeskoj Literatury, Moskva (1956);
   S. R. DE GROOT, *Thermodynamik irreversibler Prozesse*, Bibliographisches Institut, Mannheim (1960).
3. S. R. DE GROOT and P. MAZUR, *Non-equilibrium thermodynamics*, North-Holland Publishing Company, Amsterdam (1961).
4. A. V. LUIKOV and YU. A. MIKHAILOV, *Teoriya perenosa energii i veshchestva*, Akad. Nauk B.S.S.R., Minsk (1959).

Résumé—"La thermodynamique en non-équilibre" peut servir de base pour un traitement macroscopique des phénomènes irréversibles. On montre que, pour un système dans lequel la conduction thermique, la diffusion, l'écoulement visqueux etc. peuvent intervenir, on peut trouver unsustème complet d'équations différentielles décrivant le comportement irréversible, à partir des lois de conservation, de l'équilibre d'entropie et des équations décrivant le phénomène.

Zusammenfassung—Die "Nicht-Gleichgewichts-Thermodynamik" kann als Grundlage für die makroskopische Behandlung irreversibler Vorgänge dienen. Für ein System, in welchem Wärmeleitung, Diffusion, zähe Strömung usw. vor sich gehen kann, wird gezeigt, wie man einen vollständigen Satz von Differentialgleichungen zur Beschreibung des irreversiblen Verhaltens findet, indem man von den zugehörigen Erhaltungssätzen, der Entropiebilanz und phänomenologischen Gleichungen ausgeht.

Аннотация—«Неравновесная термодинамика» может служить основой для макроскопического исследования необратимых явлений переноса. Для системы, в которой имеют место явления переноса: теплопроводность, диффузия, вязкое течение и другие, выведена полная система дифференциальных уравнений, описывающая все виды переноса в их взаимосвязи. Эта система уравнений получена на основе законов сохранения, уравнения баланса энтрении и феноменологических уравнений.

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ANALYTICAL SOLUTIONS OF A SET OF EQUATIONS OF HEAT AND MASS TRANSFER FOR A SEMI-BOUNDED MEDIUM WITH DIFFERENT BOUNDARY CONDITIONS

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Аннотация—В докладе сформулированы две задачи из аналитической теории переноса тепла и массы. Аналитические решения даны с помощью комплексного применения интегральных преобразований Фурье и Лапласа.

NOMENCLATURE

α, β, γ, variables of integration over space co-ordinates;
τ, variable of integration over time;
x, y, z, co-ordinates of a point in a three-dimensional space;
t, time;
ξ, η, ζ, transformed co-ordinates x, y, z in an image region;
P, parameter of the Laplace integral transformation over time t;
$\doteq$, sign of transition from an original to an image.

THE present paper deals with an analytical theory of heat and mass transfer of a bound matter for a semi-bounded medium with boundary conditions of the first and second types.

In Luikov's papers the analytical theory of internal heat and mass transfer in capillary porous bodies is systematically developed, and is described by a set of partial differential equations of the parabolic type. This set in potential form may be represented as follows:

$$\left.\begin{aligned} \frac{\partial u}{\partial t} &= a_1^2 \nabla^2 u + a_2^2 \nabla^2 T, \\ \frac{\partial T}{\partial t} &= a_3^2 \nabla^2 T + a_4 \frac{\partial u}{\partial t}, \end{aligned}\right\} \quad (1)$$

where

$\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2} = \Delta$—is the Laplace operator;
$u(x, y, z, t)$—is the distribution function of a bound matter (moisture), kg/kg;
$T(x, y, z, t)$—is the potential function of temperature distribution, deg;
a_1^2—is the conductivity potential coefficient of the mass of bound matter, m²/hr;
a_3^2—is the heat conductivity potential coefficient (thermal diffusivity coefficient) m²/hr, $a_2^2 = a_1^2 \delta$;
δ—is the Soret coefficient, deg^{-1}.

Many problems in the analytical theory of heat and mass transfer phenomenon of bound matter are reduced to the solution of the set of equations (1) for different boundary conditions. Therefore the accumulation of solutions of system (1) at different initial and boundary conditions has considerable theoretical and practical interest.

Solutions for system (1) at boundary conditions of the first and second kinds for a semi-bounded one-dimensional and multi-dimensional medium were found by the author. These solutions are given below.

Problem 1. Find functions $u(x, y, z, t)$, $T(x, y, z, t)$ in the region

$$\Omega \begin{cases} 0 \leqslant x < \infty \\ -\infty < y < \infty \\ -\infty < z < \infty \end{cases} \quad t > 0,$$

satisfying system (1) and boundary conditions of the second kind:

$$u(x, y, z, 0) = f_1(x, y, z), \quad T(x, y, z, 0) = f_2(x, y, z); \tag{2}$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = \varphi_1(y, z, t), \quad \left.\frac{\partial T}{\partial x}\right|_{x=0} = \varphi_2(y, z, t). \tag{3}$$

Problem 2. Solve system (1) in the region of Ω with boundary conditions of the first kind:

$$u(x, y, z, 0) = f_1(x, y, z), \quad T(x, y, z, 0) = f_2(x, y, z); \tag{4}$$

$$u(0, y, z, t) = \psi_1(y, z, t), \quad T(0, y, z, t) = \psi_2(y, z, t). \tag{5}$$

Functions $u(x, y, z, t)$, $T(x, y, z, t)$ are sought in the class for which the Fourier transformations in space co-ordinates and the Laplace transformation in time, t, in the region of Ω are applicable. We start from the regularity of the functions being determined at infinity $u(x, y, z, t) \to 0$, $T(x, y, z, t) \to 0$ at $r^2 = x^2 + y^2 + z^2 \to \infty$.

The application of integral transformation methods in solving system (1) makes it possible to divide effectively the functions u and T and to carry out the solution of the problem to completion.

Let

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x, y, z, t) \exp\left[i(\eta y + \zeta z)\right] dy\, dz = v(x, \eta, \zeta, t),$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T(x, y, z, t) \exp\left[i(\eta y + \zeta z)\right] dy\, dz = T(x, \eta, \zeta, t),$$

then

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \exp\left[i(\eta y + \zeta z)\right] dy\, dz = -(\eta^2 + \zeta^2)\, v(x, \eta, \zeta, t),$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) \exp\left[i(\eta y + \zeta z)\right] dy\, dz = -(\eta^2 + \zeta^2)\, T(x, \eta, \zeta, t).$$

Denote

$$\sqrt{\left(\frac{2}{\pi}\right)}\int_0^{\infty} v(x, \eta, \zeta, t) \cos x\xi\, dx = v^*(\xi, \eta, \zeta, t),$$

$$\sqrt{\left(\frac{2}{\pi}\right)}\int_0^{\infty} T(x, \eta, \zeta, t) \cos x\, \xi\, dx = T^*(\xi, \eta, \zeta, t).$$

Applying the cosine transformation to the second derivatives

$$\frac{\partial^2 v(x, \eta, \zeta, t)}{\partial x^2}, \quad \frac{\partial^2 T(x, \eta, \zeta, t)}{\partial x^2}$$

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with regard to transformed boundary conditions (3)

$$\Phi_k(\eta, \zeta, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi_k(y, z, t)\exp\left[i(\eta y + \zeta z)\right] \mathrm{d}y\,\mathrm{d}z \quad (k = 1, 2) \tag{3*}$$

we obtain

$$\sqrt{\left(\frac{2}{\pi}\right)}\int_0^{\infty}\frac{\partial^2 v}{\partial x^2}\cos x\,\xi\,\mathrm{d}x = -\,\xi^2 v^*(\xi, \eta, \zeta, t) - \sqrt{\left(\frac{2}{\pi}\right)}\,\Phi_1(\eta, \zeta, t),$$

$$\sqrt{\left(\frac{2}{\pi}\right)}\int_0^{\infty}\frac{\partial^2 T}{\partial x^2}\cos x\,\xi\,\mathrm{d}x = -\,\xi^2 T^*(\xi, \eta, \zeta, t) - \sqrt{\left(\frac{2}{\pi}\right)}\,\Phi_2(\eta, \zeta, t).$$

Apply to system (1) the bilateral integral Fourier transformation in variables, y, z, and the cosine transformation in the variable, x. Then, taking into account condition (3*) and the regularity of functions at infinity, we obtain:

$$\left.\begin{aligned}\frac{\partial v^*}{\partial t} &= -\,a_1^2(\xi^2 + \eta^2 + \zeta^2)v^* - a_2^2(\xi^2 + \eta^2 + \zeta^2)T^* - \sqrt{\left(\frac{2}{\pi}\right)}\,(a_1^2\Phi_1 + a_2^2\Phi_2),\\ \frac{\partial T^*}{\partial t} &= -\,a_3^2(\xi^2 + \eta^2 + \zeta^2)T^* + a_4\frac{\partial v^*}{\partial t} - a_3^2\sqrt{\left(\frac{2}{\pi}\right)}\,\Phi_2\end{aligned}\right\} \tag{1*}$$

For the system (1*) the initial conditions will be:

$$F^*(\xi, \eta, \zeta) = \sqrt{\left(\frac{2}{\pi}\right)}\int_0^{\infty} F_k(x, \eta, \zeta)\cos x\,\xi\,\mathrm{d}x \ldots, \tag{2*}$$

where

$$F_k(x, \eta, \zeta) = \sqrt{\left(\frac{2}{\pi}\right)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_k(x, y, z)\exp\left[i(\eta y + \zeta z)\right] \mathrm{d}y\,\mathrm{d}z.$$

Assume

$$\bar{v}^*(\zeta, \eta, \xi, P) = \int_0^{\infty} v^*(\xi, \eta, \zeta, t)\exp(-Pt)\,\mathrm{d}t,$$

$$\bar{T}^*(\xi, \eta, \zeta, P) = \int_0^{\infty} T^*(\xi, \eta, \zeta, t)\exp(-Pt)\,\mathrm{d}t.$$

Then after the Laplace integral transformation system (1*) with due regard for the initial conditions (2*) will be as follows:

$$\left.\begin{aligned}&(P + a_1^2\rho^2)\bar{v}^*(\xi, \eta, \zeta, P) + a_2^2\rho^2\bar{T}^*(\xi, \eta, \zeta, P)\\ &\qquad = -\sqrt{\left(\frac{2}{\pi}\right)}\,[a_1^2\bar{\Phi}_1(\eta, \zeta, P) + a_2^2\bar{\Phi}_2(\eta, \zeta, P)] + F_1^*(\xi, \eta, \zeta),\\ &-a_4 P\bar{v}^*(\xi, \eta, \zeta, P) + (a_3^2\rho^2 + P)\bar{T}^*(\xi, \eta, \zeta, P)\\ &\qquad = -\sqrt{\left(\frac{2}{\pi}\right)}\,a_3^2\bar{\Phi}_2(\eta, \zeta, P) - a_4 F_1^*(\xi, \eta, \zeta) + F_2^*(\xi, \eta, \zeta),\\ &\qquad (\rho^2 = \xi^2 + \eta^2 + \zeta^2)\end{aligned}\right\} \tag{1**}$$

$$D(\xi, \eta, \zeta, P) = \begin{vmatrix}(P + a_1^2\rho^2) & a_2\rho_2\\ -a_4 P & (P + a_3^2\rho^2)\end{vmatrix} = P^2 + P(a_1^2 + a_3^2 + a_2^2 a_4)\rho^2 + a_1^2 a_3^2\rho^4$$

Denote by $P_1 = -a\rho^2$, $P_2 = -b\rho^2$ the roots of the determinant $D(\xi, \eta, \zeta, P)$ with respect to the parameter, P.

The analysis of these roots shows that the values a, b are positive. This means that we may assume $a = c_1^2$; $b = c_2^2$.

From system (1**) the transforms of potential distributions of mass and temperature may be written in the form:

$$\bar{v}^*(\xi, \eta, \zeta, P) = \frac{D_1(\xi, \eta, \zeta, P)}{D(\xi, \eta, \zeta, P)}, \quad \bar{T}^*(\xi, \eta, \zeta, P) = \frac{D_2(\xi, \eta, \zeta, P)}{D(\xi, \eta, \zeta, P)}.$$

$$D_1(\xi, \eta, \zeta, P) = \begin{vmatrix} -\sqrt{\left(\frac{2}{\pi}\right)}\,(a_1^2\bar{\Phi}_1 + a_2^2\bar{\Phi}_2) + F_1^* & a_2^2\rho^2 \\ -a_3^2\sqrt{\left(\frac{2}{\pi}\right)}\,\bar{\Phi}_2 + F_2^* - a_4F_1^* & (P + a_3^2\rho) \end{vmatrix},$$

$$D_2(\xi, \eta, \zeta, P) = \begin{vmatrix} (P + a_1^2\rho^2) & -\sqrt{\left(\frac{2}{\pi}\right)}\,(a_1^2\bar{\Phi}_1 + a_2^2\bar{\Phi}_2) + F_1^* \\ -a_4P & -a^2\sqrt{\left(\frac{2}{\pi}\right)}\,\bar{\Phi}_2 + F_2^* - a_4F_1^* \end{vmatrix}.$$

The transform of mass distribution of a bound substance in the expanded form will be,

$$\bar{v}^*(\xi, \eta, \zeta, P) = \frac{-(P + a_3^2\rho^2)\sqrt{\left(\frac{2}{\pi}\right)}\,\bar{\Phi}(\eta, \zeta, P)}{D(\xi, \eta, \zeta, P)} + \frac{a_3^2a_2^2\sqrt{\left(\frac{2}{\pi}\right)}\,\bar{\Phi}_2(\eta, \zeta, P)}{D(\xi, \eta, \zeta, P)}$$

$$+ \frac{(P + a_3^2\rho^2)\,F_1^*(\xi, \eta, \zeta, P) + \rho^2F^*(\xi, \eta, \zeta)}{D(\xi, \eta, \zeta, P)},$$

where

$$\bar{\Phi}(\eta, \zeta, P) = a_1^2\bar{\Phi}_1 + a_2^2\bar{\Phi}_2, \quad F^*(\xi, \eta, \zeta) = a_2^2(a_4F_1^* - F_2^*).$$

For the parameter transition from the image to the original by the parameter t, we use the formula for the determination of the original when the image is a normal fractional-rational function with poles of the first order:

$$\frac{Q_m(P)}{R_n(P)} = \sum_{k=1}^{n} \frac{Q_m(P_k)}{R_n'(P_k)} \exp(P_kt), \qquad (+)$$

where P_k are simple roots of integral function $R_n(P)$, $n > m$.

Using this formula and the corresponding theorem on convolutions we, in our case, obtain:

$$v^*(\xi, \eta, \zeta, t) = \sum_{k=1}^{2} a_k \left\{ \sqrt{\left(\frac{2}{\pi}\right)} \int_0^t \left[\frac{a_3^2 \cdot a_2^2}{a_3^2 - c_k^2}\,\Phi_2(\eta, \zeta, \tau) - \Phi(\eta, \zeta, \tau) \right] \right.$$

$$\left. \times \exp[-\rho^2c_k^2(t - \tau)]\,d\tau + \left[F_1^*(\xi, \eta, \zeta) + \frac{F^*(\xi, \eta, \zeta)}{a_3^2 - c_2^k} \right] \exp(-\rho^2c_k^2t) \right\},$$

where

$$a_k = \frac{a_3^2 - c_k^2}{a_1^2 + a_3^2 + a_2^2a_4 - 2c_k^2}.$$

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Subject the function $v^*(\xi, \eta, \zeta, t)$ in variable, ξ, to an inverse Fourier cosine-transformation. Then we get:

$$v(x, \eta, \zeta, t) = \sum_{k=1}^{2} \frac{2a_k}{\pi} \left\{ \int_0^t \left[\frac{a_2^2 a_3^2}{a^2 - c_k^2} \Phi_2(\eta, \zeta, \tau) - \Phi(\eta, \zeta, \tau) \right] \exp\left[-(\eta^2 + \zeta^2)(t - \tau) c_k^2\right] d\tau \right.$$

$$\times \int_0^\infty \exp\left[-\zeta^2 c_k^2 (t - \tau) \cos x\zeta \, d\zeta\right] + \int_0^\infty \left[F_1(\alpha, \eta, \zeta) + \frac{F(\alpha, \eta, \zeta)}{a_3^2 - c_k^2} \right] \exp\left[-c_k^2(\eta^2 + \zeta^2)t\right] . \, d\alpha$$

$$\left. \times \int_0^\infty \exp\left(-\zeta^2 . c_k^2 t\right) \cos \alpha\zeta \cos x\zeta . \, d\zeta \right\}.$$

Using the value of the improper integral

$$\int_0^\infty \exp\left(-a^2x^2\right) \cos(2bx) . \, dx = \frac{\sqrt{\pi}}{2a} \exp\left(-\frac{b^2}{a^2}\right)$$

and the equality $2 \cos \alpha\zeta . \cos \zeta = \cos \zeta(x + \alpha) + \cos \zeta(x - \alpha)$ we write:

$$v(x, \eta, \zeta, t) = \sum_{k=1}^{2} \frac{a_k}{c_k \sqrt{\pi}} \left\{ \int_0^t \left[\frac{a_3^2 . a_2^2}{a_3^2 - c_k^2} \Phi_2(\eta, \zeta, \tau) - \Phi(\eta, \zeta, \tau) \right] \right.$$

$$\times \frac{\exp\left[-(\eta^2 + \zeta^2)(t - \tau) c_k^2\right]}{\sqrt{(t - \tau)}} \cdot \exp\left[\frac{x^2}{4c_k^2(t - \tau)}\right] d\tau + \frac{1}{2\sqrt{t}} \int_0^\infty \left[F_1(\alpha, \eta, \zeta) \right.$$

$$\left. \left. + \frac{F(\alpha, \eta, \zeta)}{a_3^2 - c_k^2} \right] \exp\left[-c_k^2(\eta^2 + \zeta^2)t\right] \left[\exp\left(-\frac{(x - \alpha)^2}{4c_k^2 t}\right) + \exp\left(-\frac{(x + \alpha)^2}{4c_k^2 t}\right) \right] d\alpha \right\}.$$

For the transition to the original in the remaining co-ordinates we use the conversion formula for a two-dimensional Fourier transformation

$$\exp\left[-(\eta^2 + \zeta^2)a^2 t\right] \doteq \frac{1}{2a^2 t} \exp\left(\frac{y^2 + z^2}{4a^2 t}\right).$$

Thus after a reverse conversion through variables η, ζ and application of theorems on convolutions the function of mass distribution of bound matter will have the form:

$$u(x, y, z, t) = \sum_{k=1}^{2} a_k \left\{ \frac{1}{4(c_k\sqrt{\pi})^3} \int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\varphi_k^*(\beta, \gamma, \tau)}{\sqrt{[(t - \tau)^3]}} \exp\left[-\frac{x^2 + (y - \beta)^2 + (z - \gamma)^2}{4c_k^2(t - \tau)}\right] d\tau \, d\beta \, d\gamma \right.$$

$$+ \frac{1}{[2\sqrt{(\pi t)}c_k]^3} \int_0^\infty \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_k^*(\alpha, \beta, \gamma) \exp\left[-\frac{(x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2}{4c_k^2 t}\right]$$

$$\left. \times \left[1 + \exp\left(-\frac{\alpha x}{c_k^2 t}\right)\right] d\alpha \, d\beta \, d\gamma \right\}. \quad (6)$$

In formula (6) the following notation is adopted

$$\varphi_k^*(y, z, t) = \frac{a_2^2 c_k}{a_3^2 - c_k^2} \varphi_2(y, z, t) - a_1^2 \varphi_1(y, z, t),$$

$$f_k^*(x, y, z) = \left(1 + \frac{a_2^2 a_4}{a_3^2 - c_k^2}\right) f_1(x, y, z) - \frac{a_2^2}{a_3^2 - c_k^2} f_2(x, y, z).$$

The transition from the image of the temperature distribution function to the original is effected by the same successive operations which were carried out in the determination of the distribution

function of moisture. On this basis omitting these transformations we may write down the final result:

$$T(x, y, z, t) = \sum_{k=1}^{2} \gamma_k \left\{ \frac{1}{4[\sqrt{(\pi)c_k}]^3} \int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\varphi_k^{**}(\beta, \gamma, \tau)}{\sqrt{(t-\tau)^3}} \exp\left[\frac{x^2 + (y-\beta)^2 + (z-\gamma)^2}{4c_k^2(t-\tau)}\right] \right.$$
$$\times \mathrm{d}\tau\,\mathrm{d}\beta\,\mathrm{d}\gamma + \frac{1}{[2\sqrt{(\pi t)c_k}]^3} \int_0^{\infty} \int_{-\infty}^{-\infty} \int_{-\infty}^{\infty} f_k^{**}(\alpha, \beta, \gamma) \exp\left[-\frac{(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2}{4c_k^2 t}\right] \quad (7)$$
$$\left. \times \left[1 + \exp\left(-\frac{\alpha x}{c_k^2 t}\right)\right] \mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}\gamma \right\};$$

where

$$\varphi_k^{**}(y, z, t) = \left[\frac{a_2^2 a_4 \cdot c_k^2}{a_1^2 - c_k^2} - a_3^2\right] \varphi_2(y, z, t) + \frac{a_4 a_1^2 c_k^2}{a_1^2 - c_k^2} \varphi_1(y, z, t);$$

$$f_k^{**}(x, y, z) = f_2(x, y, z) - \frac{a_1^2 a_4}{a_1^2 - c_k^2} f_1(x, y, z);$$

$$\gamma_k = \frac{a_1^2 - c_k^2}{a_1^2 + a_3^2 + a_2^2 a_4 - 2c_k^2}.$$

Formulas (6) and (7) represent a solution to the first problem with very general boundary conditions.

It is known from the theory of a non-stationary heat and mass transfer process that the Soret coefficient has a very small value, so that the second term in system (1) may in some cases be neglected. When $a_2^2 = 0$ formulas (6) and (7) after certain simplifications will be reduced to:

$$u_0(x, y, z, t) = \frac{1}{[2\sqrt{(\pi t)a_1}]^3} \int_0^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_1(\alpha, \beta, \gamma) \exp\left[\frac{(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2}{4a_1^2 t}\right]$$
$$\times \left[1 + \exp\left(-\frac{\alpha x}{a_1^2 t}\right)\right] \mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}\gamma - \frac{1}{4\sqrt{(\pi^3)a_1}} \int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\varphi_1(\rho, \gamma, t)}{\sqrt{[(t-\tau)^3]}} \quad (6^*)$$
$$\times \exp\left[-\frac{x^2 + (y-\beta)^2 + (z-\gamma)^2}{4a_2^2(t-\tau)}\right] \mathrm{d}\tau\,\mathrm{d}\beta\,\mathrm{d}\gamma.$$

$$T_0(x, y, z, t) = \frac{1}{4a_3\sqrt{(\pi^3)}} \int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\varphi(\beta, \gamma, \tau)}{\sqrt{[(t-\tau)^3]}} \exp\left[-\frac{x^2 + (y-\beta)^2 + (z-\gamma)^2}{4a_3^2(t-\tau)}\right] \mathrm{d}\tau\,\mathrm{d}\beta\,\mathrm{d}\gamma$$
$$- \frac{a_4 a_1}{4(a - a^2)\sqrt{(\pi^3)}} \int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\varphi_1(\beta, \gamma, \tau)}{\sqrt{[(t-\tau)^3]}} \exp\left[-\frac{x^2 + (y-\beta)^2 + (z-\gamma)^2}{4a_3^2(t-\tau)}\right] \mathrm{d}\tau\,\mathrm{d}\beta\,\mathrm{d}\gamma$$
$$+ \frac{1}{[2a_3\sqrt{(\pi t)}]^3} \int_0^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\alpha, \beta, \gamma) \exp\left[-\frac{(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2}{4a_3^2 t}\right] \quad (7^*)$$
$$\times \left[1 + \exp\left(-\frac{\alpha v}{a_3^2 t}\right)\right] \mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}\gamma + \frac{a_4}{a_4(a_1^2 - a_3^2)\,[2\sqrt{(\pi t)}]^3} \int_0^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_1(\alpha, \beta, \gamma)$$
$$\times \exp\left[-\frac{(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2}{4a_1^2 t}\right] \times \left[1 + \exp\left(-\frac{\alpha x}{a_1^2 t}\right)\right] \mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}\gamma.$$

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where

$$\varphi(y, z, t) = \frac{a_4 a_1^2}{a_1^2 - a_3^2}\, \varphi_1(y, z, t) - \varphi_2(y, z, t);$$

$$f(x, y, z) = f_2(x, y, z) - \frac{a_1^2 a_4}{a_1^2 - a_3^2}\, f_1(x, y, z).$$

Formula (6*) represents the known classical solution of a thermal conductivity equation.

Consider a particular case of the given problem when the boundary conditions are constant values.

Let

$$f_1(x, y, z) = u_0 = \text{const.}, \qquad f_2 = (x, y, z) = \theta = \text{const.},$$

$$\varphi_1(y, z, t) = q_1 = \text{const.}, \qquad \varphi_2 = (y, z, t) = q_2 = \text{const.}$$

Then formula (6*) after some simplifications will be:

$$u_0(x, y, z, t) = u_0 + q_1 x \left[\operatorname{erf}(\alpha) - \frac{\exp(-\alpha^2)}{\alpha\sqrt{\pi}} \right] \ldots, \tag{8}$$

where

$$\alpha = \frac{x}{2a_1\sqrt{t}}; \quad \operatorname{erf}(\alpha) = 1 - \operatorname{erf}(\alpha) = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\alpha} \exp(-z^2)\, \mathrm{d}z.$$

Substituting the values u_0, θ, q_1, q_2 into formula (7*) we obtain:

$$\left.\begin{aligned} T_0(x, y, z, t) = \theta + q^* x \left[\frac{\exp(-\alpha_1^2)}{\alpha_1\sqrt{\pi}} - \operatorname{erf}(\alpha_1) \right] - \frac{a_4 \cdot a_1^2 \cdot q_1 \cdot x}{a_1^2 - a_3} \left[\frac{\exp(-\alpha_3^2)}{\alpha_3\sqrt{\pi}} - \operatorname{erf}(\alpha_3) \right] \\ \alpha_1 = \frac{x}{2a_3\sqrt{t}}, \qquad \alpha_3 = \frac{x}{2a_3\sqrt{t}}, \qquad q^* = \left[\frac{a_1^2 a_4 q_1}{a_1^2 - a_3^2} - q_2 \right]. \end{aligned}\right\} \tag{9}$$

The solution of the second problem is carried out by the same methods as used in the first one.

The transition from the required functions $u(x, y, z, t)$, $T(x, y, z, t)$ to their images is effected by the following successive integral transformations: double integral Fourier transformation by variables y, z, sine-transformation by x and the Laplace transformation by time t. As a result of the application of these transformations to system (1) and to boundary conditions (4) and (5) we find the solution of a set of differential equations of internal heat and mass transfer in the images.

The transition from the image to the original is effected by reverse integral transformations They are carried out in inverse order.

Omitting all these transformations we may write down the result of the second problem considered:

$$\left.\begin{aligned} u(x, y, z, t) = \sum_{k=1}^{2} \frac{1}{[2\sqrt{(\pi)}c_k]^3} \left\{ \frac{x}{c_k^2} \int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{[m_k\psi_1(\beta,\gamma,\tau) - n_k\psi_2(\beta,\gamma,\tau)]}{\sqrt{[(t-\tau)^5]}} \right. \\ \times \exp\left[-\frac{x^2 + (y-\beta)^2 + (z-\gamma)^2}{4c_k^2(t-\tau)} \right] \mathrm{d}\tau\, \mathrm{d}\beta\, \mathrm{d}\gamma + \frac{1}{\sqrt{(t^3)}} \int_0^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \\ \times [\nu_k f_1(\alpha, \beta, \gamma) - \mu_k f_2(\alpha, \beta, \gamma)] \cdot \exp\left[-\frac{(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2}{4c_k^2 t} \right] \\ \left. \times \left[1 - \exp\left(-\frac{\alpha x}{c_k^2 t} \right) \right] \mathrm{d}\alpha\, \mathrm{d}\beta\, \mathrm{d}\gamma \right\}. \end{aligned}\right\} \tag{10}$$

$$T(x, y, z, t) = \sum_{k=1}^{2} \frac{1}{[2\sqrt{(\pi)c_k}]^3} \left\{ \frac{x}{c_k^2} \int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{[\beta_k \psi_2(\beta, \gamma, \tau) - \gamma_k \psi_1(\beta, \gamma, \tau)]}{\sqrt{[(t-\tau)^5]}} \right.$$
$$\times \exp\left[-\frac{x^2 + (y-\beta)^2 + (z-\gamma)^2}{4c_k^2(t-\tau)} \right] d\tau\, d\beta\, d\gamma + \frac{1}{\sqrt{(t^3)}} \int_0^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [\lambda_k f(\alpha, \beta, \gamma) - x_k f_2(\alpha, \beta, \gamma)]$$
$$\left. \times \exp\left[-\frac{(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2}{4c_k^2 t} \right] \left[1 - \exp\left(\frac{\alpha x}{c_k^2 t} \right) \right] d\alpha\, d\beta\, d\gamma \right\}, \tag{11}$$

where

$$m_k = A_k a_1^2 (a_3^2 - c_k^2); \quad n_k = A_k a_2^2 c_k^2; \quad \nu_k = 1 - A_k (a_1^2 - c_k^2); \quad \mu_k = A_k a_2^2;$$
$$\beta_k = A_k [(a_1^2 - c_k^2) a_3^2 - a_2^2 a_4 c_k^2]; \quad \gamma_k = A_k a_1^2 a_4 c_k^2; \quad \lambda_k = A_k (a_1^2 - c_k^2);$$
$$\chi_k = a_1^2 a_4 A_k; \quad A_k = (a_1^2 + a_3^2 + a_2^2 a_4 - 2c_k^2)^{-1}.$$

It should be noted that for the transition from the image to the original by parameter ξ it is necessary to use the known formula of the improper integral

$$\int_0^{\infty} \xi \exp(-a^2 \xi^2 t) \sin x\, \xi\, d\xi = \frac{x\sqrt{\pi}}{4a^3 \sqrt{(t^3)}} \exp\left(-\frac{x^2}{4a^2 t} \right). \tag{12}$$

With the help of a similar method the author has solved boundary problems of the first and the second kind for a system of "n" non-uniform differential equations of a parabolic type.

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REFERENCES

1. A. V. Luikov, *Theory of energy and substance transfer* (*Teoriya perenosa energii i veschestva*). Izdaniie Akad. Nauk BSSR, Minsk (1959).
2. P. V. Tsoi, *Inzh.-Fiz. Zh.*, No. 6, 112 (1960); No. 12, 53 (1960); No. 1, 120 (1961); No. 4, 69 (1961).

Abstract—Two problems for the system of equations of heat and mass transfer are considered. The Laplace–Fourier integral transformations are used for the solution.

Résumé—Deux cas de système d'équations de transport de chaleur et de masse sont considérés. Pour les résoudre on utilise les transformations d'intégrales de Fourier et de Laplace.

Zusammenfassung—Die Gleichungssysteme für Wärme- und Stoffübertragung werden auf zwei Problemstellungen angewandt. Zur Lösung dienten die Laplace–Fourier-Integraltransformationen.

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TURBULENT HEAT TRANSFER IN A FLOW OF LIQUID METALS

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Аннотация—Работа посвящена экспериментальному исследованию турбулентного переноса тепла в потоке жидких металлов.

Проведено одновременное измерение температурного поля в потоке и коэффициента теплоотдачи методами, учитывающими термическое контактное сопротивление на границе раздела фаз.

При этом удалось достаточно чётко разделить два явления, определяющих передачу тепла в жидких металлах: совместный молекулярный и турбулентный перенос тепла, описываемый полуэмпирической теорией теплообмена, и имеющее место в жидких металлах термическое контактное сопротивление на поверхости теплообмена, вызываемое выпаданием окислов жидких металлов в районе ламинарного подслоя и другими факторами.

Проведенные исследования показали, что при учёте термического контактного сопротивления полученные экспериментальные данные хорошо совпадают с расчётными, основанными на полуэмпирической теории теплообмена.

NOMENCLATURE

r_0, tube radius;

$\xi = \frac{r}{r_0}$, relative co-ordinate of point;

ϵ_a, turbulent heat transfer coefficient;

ϵ_j, turbulent momentum transfer coefficient;

R_c, thermal contact resistance;

t_w, wall temperature;

q, local heat flow;

q_w, heat flow at wall;

λ, heat conductivity;

u, flow velocity at given point;

w, characteristic velocity.

In their present state of development, turbulent theories do not permit an analytical determination of turbulent heat transfer in a liquid flow. Semi-empirical heat transfer theories based on the analogy between heat and momentum transfer are therefore widely used. Using various assumptions, the authors calculated turbulent heat transfer, determined the temperature field in a liquid flow and the heat transfer coefficients. The validity of assumptions in semi-empirical theories may be verified by experiments on measurement of temperature fields in liquid metals. Because of the high thermal conductivity of liquid metals the main temperature drop is not just concentrated within the thin laminar sub-layer as in ordinary low heat-conducting fluids but it extends into the turbulent bulk flow. This allows sufficiently accurate experimental determination of temperature gradients across the section of a tube and reliable verification of assumptions for semi-empirical heat transfer theories.

Martinelli was the first to apply the theory of the hydrodynamic analogy to liquid metals taking account of molecular heat conduction in a turbulent main stream. The assumption was made in calculations that the ratio of turbulent heat and momentum transfer coefficients ($\epsilon = \epsilon_a/\epsilon_\nu$) does not depend on the radius and velocity of the flow. Lyon obtained the general equation for the heat transfer coefficient in a tube:

$$\frac{1}{Nu} = 2\int_0^1 \frac{\int_0^1 \frac{u}{w}\,\xi\,\mathrm{d}\xi}{1 + \epsilon\,\frac{\epsilon_\nu}{\nu}\,p_\eta}\,\mathrm{d}\xi \qquad (1)$$

where

$$\xi = r/r_0.$$

Using Martinelli's main assumptions he presented the calculation results for $\epsilon = 1$ as a formula

$$Nu = 7 + 0{\cdot}025\, Pe^{0{\cdot}8}. \qquad (2)$$

The majority of experimental data on heat transfer to liquid metals have been obtained under conditions where the percentage of oxides and purity of the heat transfer surface was not under control and is noticeably below the values calculated by formula (2). Before temperature fields were measured reliably in liquid metal flow, the opinion existed that the reason for divergence between experimental data and the Martinelli–Lyon theory lay in the non-validity of the assumption that $\epsilon = 1$. Using the high thermal conductivity of liquid metals as a basis, Voskresensky, Deissler *et al.* made an analytical attempt to take into account heat transfer by conduction from separate moles into the surrounding medium. The value of ϵ turned out to be noticeably less than unity. In determining ϵ Voskresensky, Likoudis and Touluk'yan tried to use experimental data on heat transfer to liquid metals; however the authors erroneously assumed that in these data there was no thermal contact resistance. This led to an unjustifiably low estimate of turbulent heat transfer and, consequently, of ϵ for liquid metals.

In our opinion, the thermal contact resistance on a heat transfer surface is one of the reasons for divergence between various experimental data and these data and formula (2). Apparently, the value of this resistance depends upon the kind and purity of the liquid metal, wall material and upon several other factors. For evaluating the role of the thermal contact resistance the heat transfer coefficient may be determined by two methods:

(1) by measuring and treating the temperature field in a flow of liquid metals,

(2) by measuring wall temperature and average temperature of the liquid metal.

On the basis of temperature fields measured in a liquid flow the turbulent heat transfer coefficient may also be determined and the value ϵ calculated for liquid metals and water.

This paper presents the results of experiments on determining the heat transfer coefficients and on measuring the temperature fields in a flow of different alkali and heavy liquid metals (sodium, mercury, etc.). The metals investigated are notable for rather a wide range of number $Pr = 0{\cdot}005$–$0{\cdot}05$. Several tests on measuring the temperature fields were carried out with water ($Pr \simeq 10$).

Experimental section I on which heat transfer to the alkali metals was investigated is horizontal. The block of a movable thermocouple was attached to the tube with a gliding fit and sealed with "sphere-in-cone connexion". The thermocouple, of an alundum wire (nichrome and constantan 0·2 mm in diameter) was inserted into a stainless capillary 0·8 × 0·15 mm in diameter. The end of the casing was sealed integrally with thermo-electrodes. The displacement of the thermocouple over the tube diameter was carried out using a joint protected by a bellows. The accuracy of displacement along the diameter was ensured by guides at the end of the thermocouple block. The location of the thermocouple in the tube was shown by a dial indicator. The calibration was carried out by a periscope PBII-457 to within 0·1 mm.

In the same section as the movable thermocouple, three nichrome–constantan thermocouples were attached to the wall of the tube. Their junctions were sealed to the wall of the tube. A heat flow was created by a nichrome heater made out of wire 3 mm in diameter and insulated from the wall by a uniform mica layer 0·4 mm in thickness. Radial heat leakage was measured and compensated for. Over the liquid metal velocity range investigated in section I the inertia forces exceeded gravitational ones.

Experimental section II on which experiments with heavy metals and water were carried out, was located vertically. The liquid flowed upwards. The temperature field was measured simultaneously with two movable thermocouples situated in one sonde; it ensured mutual control in the readings. The thermocouples were made of alundum thermo-electrode chromel–alumel wire 0·1 mm in diameter. The cover was 20 mm in thickness.

Two thermo-electrodes were placed in one capillary 0·5 × 0·1 mm in diameter in stainless

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steel IXI8H9T. The end of the capillary was sealed together with the thermo-electrodes forming a hot junction of the thermocouple. In experiments with water the junction of the thermocouple 0·2 mm in diameter was open. The displacement of the thermocouples along the diameter was carried out by two reversible electric motors. The accuracy of displacement was secured by the guides. The location of the thermocouples in the tube was determined by a revolution counter within 0·01 mm. In the same section where the movable thermocouples travelled two other similar thermocouples were attached to the wall of the tube. Their junctions were sealed to the tube wall. The heat flow was created by an electric heater made out of nichrome band 6 × 0·2 mm in section and insulated from the tube with a uniform micalex layer 0·3 mm in thickness. Radial heat leakage was compensated for by an external heater. Main characteristics of the experimental sections are given in Table 1.

Table 1. *Leading particulars of experimental sections*

Name		Section I	Section II
Tube material			Steel IXI8H9T
External tube diameter (D) (mm)		42	34
Internal tube diameter (d, $2ra$) (mm)		31·1	29·3
Length of heat transfer section (mm)	l_h	1194	980
	l_h/d	38	34
Distance from beginning of heating section to thermocouples (mm)	$l_{th'}$	976	945
	$l_{th'}/d$	31	32
Distance from tube entry to thermocouples (mm)	l	1166	985
	l/d	37	34

When testing the average temperature of the liquid metal both the entrance to and exit from the experimental section were measured. All the thermocouples were graduated by calibration platinum platinum–rhodium thermocouples. The measurement of electric power was carried out by astatic wattmeters of the 0·2 and 0·5 grades. Liquid metal rate was measured by magnetic and restrictive flow meters. When testing the alkali metals they were subjected to continuous purification from oxides in a cold trap. Oxygen percentage was within the range of 0·02–0·05 per cent by weight. Heavy metals were not subjected to continuous purification. At the beginning of the experiment the oxygen percentage was $\sim 1 \times 10^{-3}$ per cent by weight and did not change during the experiment.

When measuring temperature fields in liquid flow, turbulent pulsations were found, the value of which attained ± 20 per cent of the value of the total temperature. In experiments it was found that the amplitude and frequency of pulsations depended on the value of the heat flow, physical properties, liquid flow regime and dimensionless distance from the wall. In liquid metals where a temperature gradient change over the tube section occurs smoothly, maximum pulsations were observed approximately in the middle region between the centre and the wall of the tube. The maximum pulsations were observed near the wall in water which has a sharp temperature gradient in the layer near the wall.

Such a change of temperature pulsations over the tube radius qualitatively agrees with the hypothesis that the value of turbulent temperature pulsations is proportional to mixing length and temperature gradient. At the same time experiments showed that the temperature pulsations on the axis and near the wall of the tube were not equal to zero. Temperature pulsations were also observed in the tube wall itself but somewhat different from those in the liquid. The liquid temperature pulsations near the wall and temperature vibrations of the wall itself denote that the heat transfer process through the liquid layer near the wall and through the heat transfer surface is not, strictly speaking, stationary (steady).

Temperature measurement in the flow of liquid metal was carried out with the help of movable thermocouples fixed at nine to twelve points with respect to the tube radius. The movable thermocouples were differentially connected with the thermocouples fixed at the entrance to the section. For reliable determination of the average temperature values indications of all the thermocouples were recorded on

diagrams of self-balancing fast potentiometers ЭПП-09 with limits of scale 0–0·5 mV connected in series with the low-ohmic potentiometer. The readings of thermocouples recorded for 30 to 50 s were averaged in time and plotted on diagrams with co-ordinates $(t - t_{u_s})$, ξ.

The wall temperature required for calculating the heat transfer coefficient was determined by extrapolation of the temperature profile on the wall. The validity of extrapolation was verified by the boundary condition:

$$\left.\frac{\mathrm{d}t_e}{\mathrm{d}r}\right|_{r=r_0} = \frac{q_w}{\lambda_e}. \tag{3}$$

The average temperature of the liquid metal flow was calculated by the formula:

$$\bar{t}_e = \frac{\int_0^{r_0} u t_e r \mathrm{d}r}{\int_0^r u r \mathrm{d}r}. \tag{4}$$

In calculations the universal law of velocity distribution was used

$$u^+ = 5{\cdot}5 + 2{\cdot}5 \ln y^+. \tag{5}$$

The application of other equations of the velocity distribution in a tube slightly alters calculation values of the average temperature of a liquid metal.

It is interesting to note that the dimensionless radius where the local temperature of a liquid is equal to the average, is the same for all the liquid metals investigated and equal to 0·7–0·75. It agrees with the theory and the results of other experimental works.

The values of *Nu* obtained by working out the temperature fields measured in a flow of liquid metals agree well both amongst themselves and with earlier results obtained with alloys of sodium, potassium and mercury. The coincidence of experimental data with Lyon formula (2) over rather a wide range of $Pe = 100$–$12\,000$ is observed. It shows that the heat transfer process without accounting for surface phenomena at the "wall–liquid metal" boundary is described by a common critical ratio (2) both for alkali and heavy liquid metals.

However the coincidence between experimental data with formula (2) does not confirm the Lyon assumption that the ratio of turbulent heat and momentum transfer coefficients $\epsilon = \epsilon_a/\epsilon_\nu$ does not change over the tube section and is equal to unity at all *Pe* numbers. As the further calculation of the obtained data on temperature fields demonstrated the ratio of turbulent heat and momentum transfer coefficients does change over the tube radius and depends on *Pe*. In the region of small *Pe* numbers the relation ϵ is somewhat less than unity, but at great *Pe* numbers, slightly exceeds it.

In experiments the heat transfer coefficient was also determined by methods taking into account the thermal contact resistance on a heat transfer surface. Considerable errors appear when measuring the mean temperature drop between a liquid metal with high heat conductivity (alkali metals) and the wall of stainless steel to which the thermocouples were attached.

To confirm the obtained values of heat transfer coefficients taking account of the thermal contact resistance, the transient method of measuring was also used. Results of experiments treated in criterial form are given in Fig. 1. Experimental data for alkali metals agree satisfactorily with formula (2); for heavy liquid metals these data are considerably below this formula.

The coincidence of data on alkali metals obtained by different methods (with and without regard to the thermal contact resistance) shows the absence of the thermal contact resistance in experiments. This fact together with the results of the work carried out earlier on an alloy of sodium and potassium permits the conclusion that in some cases after thorough purification of alkali metals from oxides it is possible to avoid thermal contact resistance altogether. For heavy metals the difference in values of heat transfer coefficients obtained by different methods shows the presence of the thermal contact resistance on the wall–liquid metal boundary. Simultaneous measurement of the temperature distribution in the wall and in the flow of liquid metal enabled the calculation to be done of the thermal contact resistance of liquid metal flow down a stainless steel pipe (IX18H9T) (18% chrome steel). It was determined by the formula:

$$R_c = \frac{t_w^{\mathrm{II}} - t_w^{\mathrm{I}}}{q_w},$$

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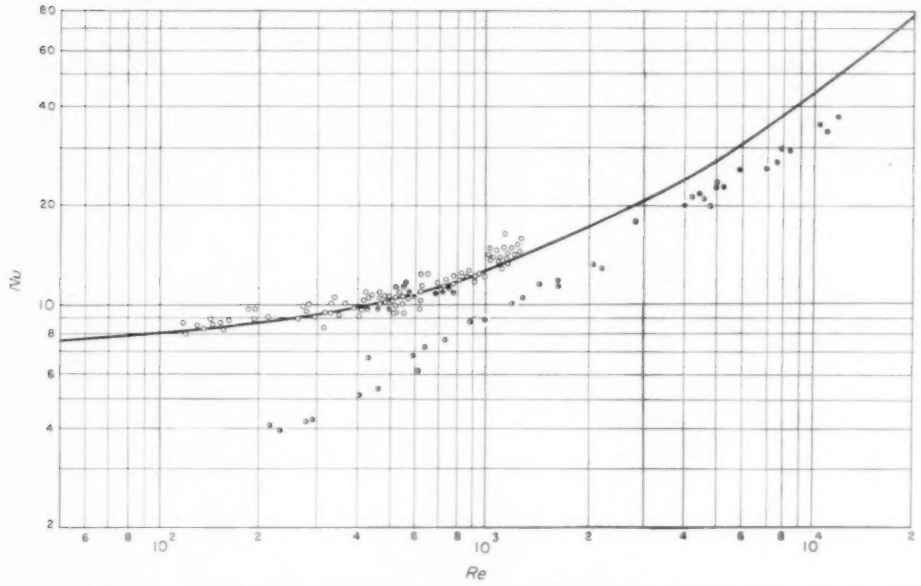

FIG. 1. Data on heat transfer to liquid metals obtained by methods taking into account thermal contact resistance.

Alkali metals:
—stationary measurement method,
—non-stationary measurement method.
—Heavy metals,
—calculation curve by formula (2).

where t_w^{II} and t_w^{I} are found by two methods (from the wall side and liquid side). The results obtained are plotted in Fig. 2. Under these conditions of testing the value of the thermal contact resistance was not kept constant and depended on the liquid metal flow rate to the same degree as the thickness of a laminar sub-layer. Hence, we may assume that the thermal contact resistance is proportional to the thickness of the laminar sub-layer, and, consequently, to the tube diameter. This conclusion is confirmed by the fact that in criterial treatment the experimental data obtained on tubes with different diameters (9–30 mm) coincide. The values of the thermal contact resistance plotted in Fig. 2 may be presented in a dimensionless form:

$$\frac{R_c\lambda_e}{d} = 6 \times 10^{-3} + \frac{1300}{Re}. \qquad (6)$$

The given formula is not, apparently, a generalized relation for all the heavy liquid metals but is valid only for the specific conditions under which experiments were carried out.

The facts obtained allow to some extent clarification of the physical meaning of the thermal contact resistance in a flow of heavy liquid metal in tubes. The deposition of a film of liquid metal oxides in the region of the laminar sub-layer is one of the reasons for the appearance of thermal contact resistance. Therefore for constant temperature of the liquid metal and for constant percentage of oxides in it, the value of the thermal contact resistance depends mainly on the same factors as does the thickness of the laminar sub-layer (upon flow rate of the liquid metal and the tube diameter). Chemical analysis showed that, in fact, the concentration of oxides near the wall was one order greater than in the bulk flow of liquid metal. The mechanism of the thermal contact resistance may differ for different liquid metals as well as for various materials for the heat transferring surface.

A simultaneous change both of temperature

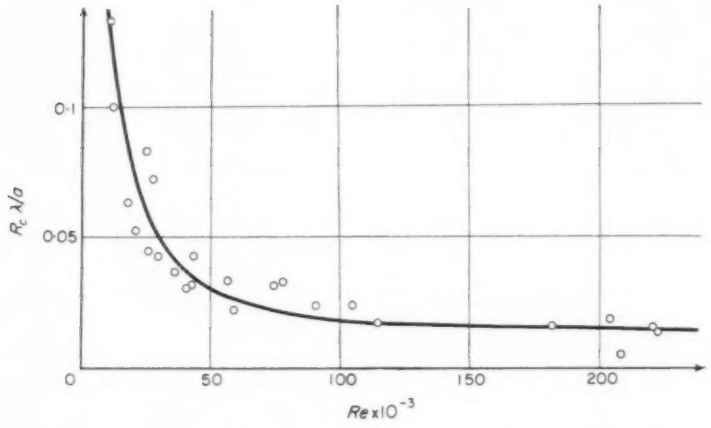

FIG. 2. Thermal contact resistance in a flow of heavy liquid metal in a tube of steel IX18H9T.

fields in the flow and heat transfer coefficients carried out by methods taking into account the thermal contact resistance allowed a fairly definite distinction to be drawn between the two processes determining heat transfer to liquid metals. The first process, bound up with molecular and turbulent transfer of heat, may be described by semi-empirical heat exchange theories. Experiments have shown such heat transfer to be described to a first approximation by the Martinelli–Lyon theory. At present the second process caused by the thermal contact resistance on a heat transfer surface cannot be estimated theoretically.

Calculation from the temperature fields measured allowed the determination of turbulent heat transfer in liquid flow. The turbulent heat transfer coefficient was derived from the equation:

$$\epsilon_a = \frac{q/q_w}{\partial t/\partial \xi}\frac{r_0 q_w}{c_p \gamma} - a. \qquad (7)$$

The ratio of local heat flux to the flow along the wall was obtained from the relations involving heat balance of an elementary liquid volume and from the application of the universal law of velocity distribution:

$$q/q_w = \frac{1}{\xi}\frac{u^*}{w}\left[(4{\cdot}25 + 2{\cdot}5 \ln y^+)\,\xi^2 - 2{\cdot}5\,(1-\xi^2)\ln\frac{r_0}{y}\right]. \qquad (8)$$

Equation (8) is valid only for the turbulent mainflow, i.e. at $y^+ \geqslant 30$.

When dealing with experimental data the determination of local temperature gradients gives the greatest difficulty. For some experimental data the following formula describing the temperature field was chosen:

$$t - t_{u_s} = A\xi^2 + B\xi^3 + C\xi^4. \qquad (9)$$

Coefficients in the series were determined by the method of least squares. The temperature gradient was determined by differentiation. The graphical differentiation of the temperature fields was carried out as well. Values of the temperature gradients obtained by analytical and graphical methods agreed well between themselves. The temperature gradients calculated by the graphical method were applied to determining ϵ_a by formula (7). In Fig. 3 the distribution of the turbulent heat transfer coefficient over the tube section is given. The coefficient ϵ_a increases both with *Re* and with distance from the wall. It should be noted that the coefficient ϵ_a is not equal to unity on the axis of the tube. However since there is considerable error when determining ϵ_a in the central region of the flow ($\xi < 0{\cdot}4$), it is difficult to speak of a precise value of this coefficient in the centre of the tube.

Heat transfer even to liquid metals possessing high heat conductivity is determined, to a

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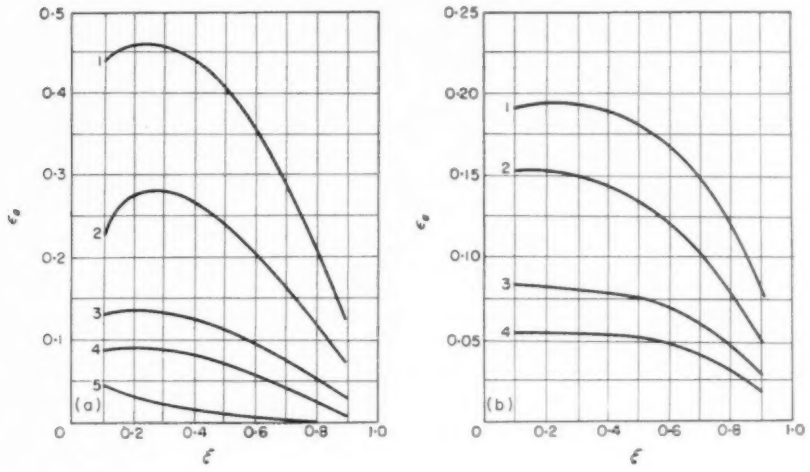

FIG. 3. Distribution of turbulent heat transfer coefficient over the tube sections at different *Re* numbers.

(a) heavy metal: $\lambda \cong 10$ kcal/m hr °C,
1—$Re = 200\,000$, 2—$Re = 120\,000$,
3—$Re = 60\,000$, 4—$Re = 38\,000$,
5—$Re = 121\,000$;

(b) alkali metal: $\lambda \cong 40$ kcal/m hr °C,
1—$Re = 24\,500$, 2—$Re = 16\,500$,
3—$Re = 11\,000$, 4—$Re = 7900$.

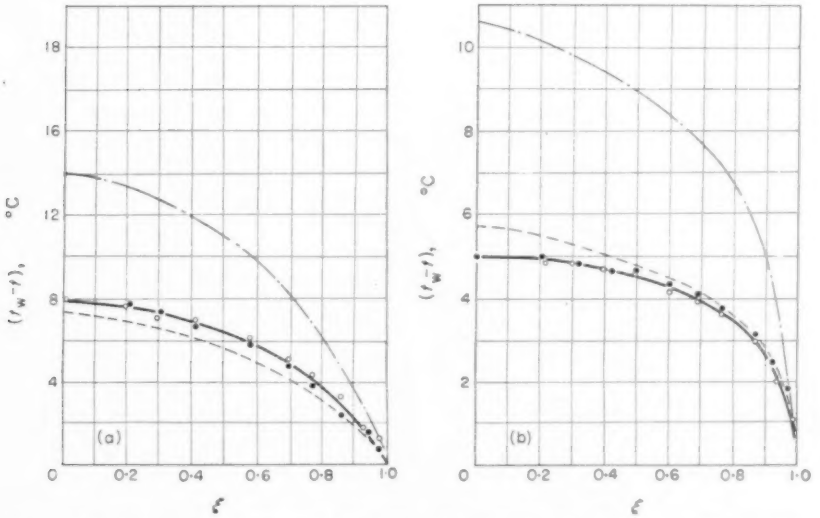

FIG. 4. Comparison between temperature fields determined experimentally in heavy liquid metal flow with calculated values.

(a) $Re = 26\,900$, $Pr = 0{\cdot}022$, $q = 18\,700$;
(b) $Re = 228\,000$, $Pr = 0{\cdot}022$, $q = 47\,600$.

——— curve averaging experimental points,
– – – – – curve calculated by the Lyon equation (1),
—·— curve calculated by the Voskresensky formula.

marked degree, by turbulent heat transfer. At large numbers of $Re \sim 200\,000$ such heat transfer plays the determining role. In his calculations Voskresensky overestimated the value of molecular heat transfer. Comparison between temperature fields determined experimentally and those calculated by the Lyon and Voskresensky theoretical relations leads to the same conclusion (Fig. 4). The temperature profile calculated by the Voskresensky theory lies considerably above the experimental points. The experimental data coincide with the temperature profile calculated from the Lyon ratio (1) more satisfactorily. At low numbers of $Re \sim 30\,000$ experimental points lie above and at large numbers of $Re \sim 200\,000$ below the Lyon curve.

Apparently, the assumption that $\epsilon = 1$ leads to overestimation of the turbulent heat transfer at low, and to underestimation at high numbers of *Re*. Such a character of change of ϵ depending on *Re* is quite understandable. Probably, at small *Re* numbers heat losses transferred by a mole due to high liquid heat conduction may be greater than momentum losses, i.e. ($\epsilon < 1$). With highly developed turbulent flow moles may appear whose momentum vanishes quicker than heat ($\epsilon > 1$). In Fig. 5 the dependence of ϵ on *Re* at $\xi = 0{\cdot}8$ is given for different liquids; liquid metals ($Pr \ll 1$) and water ($Pr \simeq 10$). The value ϵ_ν was calculated by the formula:

$$\epsilon_\nu = \frac{u^*(1 - y/r_0)}{2{\cdot}5} - \nu. \tag{10}$$

Within the accuracy of experiments we may assume that the *Pr* number hardly influences the coefficient ϵ. Probably, the mechanism of turbulent heat transfer for different liquids differs only slightly in the different cases and is mainly determined by the hydrodynamics of the flow. Experiments showed that the value ϵ changes with the increase in the distance from the wall, however, in the region which determines heat transfer ($\xi = 0{\cdot}5$–$0{\cdot}9$) this value appeared to be approximately constant.

Experimental values of the coefficient ϵ_ν were used for determining *Nu* by formula (1). The *Nu* numbers determined by this method are described by formula (2) satisfactorily and coincide well with the values obtained by calculating the average temperature drop from the temperature profile.

The investigation on temperature fields in a flow of liquid metals allows experimental determination to be done of the value of turbulent heat transfer coefficients and confirmation of the validity of assumptions of semi-empirical heat transfer theories, but we still are unable to improve on our knowledge of the heat transfer process itself. Apparently, only a thorough analysis of turbulent temperature and velocity pulsations will make it possible to develop a fundamental heat transfer theory.

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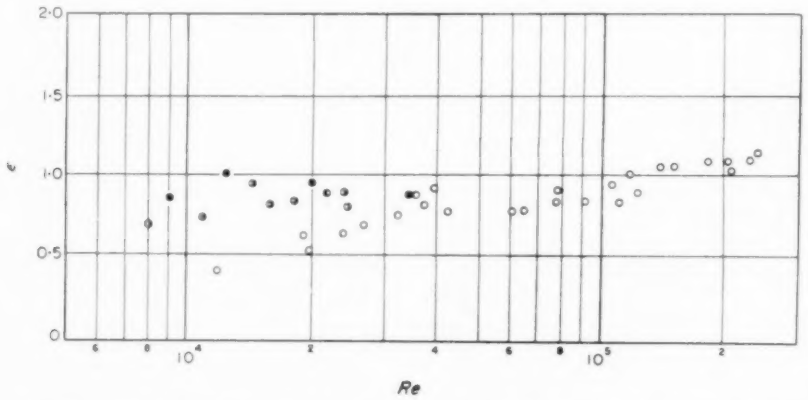

FIG. 5. Dependence of ϵ on *Re* at $\xi = 0{\cdot}8$ for different liquids ●—water ($Pr \sim 10$); ◑—alkali metal ($Pr \sim 0{\cdot}05$); ○—heavy metal ($Pr \sim 0{\cdot}02$).

Abstract—The present paper deals with experimental investigations into turbulent heat transfer in a flow of liquid metals.

Measurement of the temperature field in the flow as well as the heat transfer coefficient is carried out by methods which take the thermal contact resistance at an interface into consideration.

Two phenomena characterizing heat transfer in liquid metals are distinguished with sufficient accuracy, i.e. combined molecular and turbulent heat transfer, described by a half-empirical heat transfer theory, and the thermal contact resistance in liquid metals on a heat transfer surface, caused by the precipitation of liquid metal oxides in the region of a laminar sublayer and by a number of other factors.

The investigations show that the experimental data agrees well with calculated data based on the half-empirical heat transfer theory when the thermal contact resistance is taken into account.

Résumé—Cet article concerne des recherches expérimentales sur la transmission de chaleur turbulente dans un écoulement de métaux liquides.

Les mesures du champ de température dans l'écoulement et du coefficient de transmission de chaleur sont effectuées selon des méthodes qui tiennent compte de la résistance thermique de contact à l'interface.

Deux phénomènes caractéristiques de la transmission de chaleur dans les métaux liquides sont mis en évidence avec une préision suffisante: d'une part, la transmission de chaleur turbulente et moléculaire, décrite par une théorie semi-empirique de la transmission de chaleur et d'autre part, la résistance thermique de contact sur une surface d'échange due au dépôt d'oxyde métallique liquide dans la région de la sous couche laminaire et à d'autres facteurs.

Les recherches montrent que les données expérimentales sont en bon accord avec les résultats du calcul fait à partir de la théorie semi-empirique de transmission de chaleur, quand on tient compte de la résistance thermique de contact.

Zusammenfassung—Der Wärmeübergang bei turbulent strömenden flüssigen Metallen wurde experimentell untersucht.

Die Messung sowohl des Temperaturfeldes in der Strömung als auch des Wärmeübergangskoeffizienten erfolgte nach Methoden, die einen thermischen Kontaktwiderstand an den Berührungsflächen berücksichtigen.

Zwei charakteristische Erscheinungen des Wärmetransports in flüssigen Metallen zeichneten sich mit genügender Genauigkeit ab: nämlich, der kombiniert molekulare und turbulente Wärmetransport, der durch eine halbempirische Wärmetransporttheorie wiederzugeben ist und der thermische Kontaktwiderstand der flüssigen Metalle an der Wärmeübergangsfläche. Dieser Kontaktwiderstand tritt als Folge der Ausscheidung von Metalloxyden in einer laminaren Unterschicht und anderen Ursachen auf.

Die Untersuchungen zeigen gute Übereinstimmung zwischen experimentellen und gerechneten Werten. Letztere wurden mit Hilfe der halbempirischen Wärmetransporttheorie unter Berücksichtigung des thermischen Kontaktwiderstandes ermittelt.

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ON POSSIBLE METHODS OF SOLVING BOUNDARY LAYER PROBLEMS IN THE PRESENCE OF DISSOCIATION AND DIFFUSION

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Аннотация—В статье приводятся уравнения движения смеси газов в ламинарном и турбулентном пограничном слоях. Даётся обзор возможных методов их решения.

В частности, указаны некоторые новые классы автомодельных решений и схема приближённого решения уравнений с использованием метода Кочина-Лойцянского.

Приводятся некоторые новые результаты по турбулентному пограничному слою.

NOMENCLATURE

A, thermal equivalent of work;
c_{pi}, heat capacity of the ith component of a mixture at a constant pressure;
c_{vi}, molar heat capacity of the ith component at constant volume;
D_i, diffusion coefficient of the ith component;
E, activation energy;
E^+, E^-, radiant energy full flows, incident and reflected;
F_x, F_y, mass forces vector component (with respect to mass unit);
f, arbitrary function of x, u and T_w;
H, total heat content of mixture mass unit;
h_i, mass unit heat content of the ith component;
i, number of mixture components;
J_y^i, diffusion vector component of the ith component;
K_i, concentration change velocity of the ith component due to chemical reactions (of non-equilibrium);
M_i molecular weight of the ith component;
m, averaging coefficient;
p, pressure;
p_∞, pressure of an adiabatically decelerated flow;
q_y, heat flow vector component;
r, number of reactions when the ith component originates;
T, temperature;
T_w, wall temperature;
u, velocity on the edge of a boundary layer;
u_i, one mole internal energy of the ith component ($=u_{i0} + \int_0^T c_{vi}\,dT$);
u_{i0}, internal energy of one mole at absolute zero;
v_x, v_y, velocity vector components of a mixture as a continuous medium;
w_α, αth reaction rate;
α, absorption coefficient;
δ, boundary layer thickness;
δ^*, displacement thickness;
δ_l, laminar sublayer thickness;
ϵ_1, volumetric heat release ($\epsilon = 0$ for a plane problem; $\epsilon = 1$ for an axisymmetric problem);
λ, mixture heat conductivity coefficient;
μ, mixture viscosity coefficient;
μ_i, viscosity of the ith component;
$\nu_{i\alpha}$, stoichiometric coefficient;
ξ_i, relative mass concentration of the ith mixture component [$\xi_i = (\rho_i/\rho)$];
ρ, mixture density;
ρ_∞, density of an adibatically decelerated flow;
σ, Boltzman constant;
σ_e, conductivity;
$\tau_{x,y}$, friction tensor component;
χ_i, molar concentration.

1. EQUATIONS OF MOTION

THE steady equations of motion of an ideal gas

mixture in the boundary layer of a wing or of axisymmetrical bodies in the presence of radiation diffusion may be written in the form:

(a) mass conservation equation:

$$\frac{\partial}{\partial x}\rho v_x r^\epsilon + \frac{\partial}{\partial y}\rho v_y r^\epsilon = 0; \tag{1.1}$$

(b) momentum equation:

$$\rho\left(v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y}\right) = -\frac{\partial p}{\partial x} + \rho F_x + \frac{\partial \tau_{xy}}{\partial y}; \tag{1.2}$$

$$0 = -\frac{\partial p}{\partial y} + \rho F_y; \tag{1.3}$$

(c) diffusion equation:

$$\rho\left(v_x\frac{\partial \xi_i}{\partial x} + v_y\frac{\partial \xi_i}{\partial y}\right) = -\frac{\partial}{\partial y}J_y^i + K_i; \tag{1.4}$$

(d) energy equation:

$$\rho\left(v_x\frac{\partial H}{\partial x} + v_y\frac{\partial H}{\partial y}\right) = A\rho(F_x v_x + F_y v_y) + \frac{\partial}{\partial y}(q_y + A v_x\tau_{xy}) + \frac{\partial}{\partial y}(E^+ - E^-) + \rho\epsilon_1; \tag{1.5}$$

(e) radiation transfer equation, which for a grey body (if radiant energy dispersion is to be neglected and local thermodynamic equilibrium is to be assumed) will have the form:

$$\frac{\partial E^+}{\partial y} = m\alpha(\sigma T^4 - E^+), \tag{1.6}$$

$$\frac{\partial E^-}{\partial y} = m\alpha(\sigma T^4 - E^-); \tag{1.7}$$

(f) equation of state

$$\frac{p}{\rho} = \frac{RT}{M},$$

where

$$H = \sum_i \xi_i\frac{u_i}{M_i} + \frac{p}{\rho} + A\frac{v_x^2}{2} = h + A\frac{v_x^2}{2};\quad \frac{1}{M} = \sum_i\frac{\xi_i}{M_i}. \tag{1.8}$$

When considering motion of gas in the absence of electric and magnetic fields, mass forces may be neglected. In the case of a conducting gaseous medium, the mass forces components ρF_x and ρF_y are determined from the equations:

$$\left.\begin{aligned}\rho F_x &= -\frac{\partial}{\partial x}\frac{\mathbf{B}_x^2 + \mathbf{B}_y^2 + \mathbf{B}_z^2}{2} + \mathbf{B}_x\frac{\partial B_x}{\partial x} + \mathbf{B}_y\frac{\partial \mathbf{B}_x}{\partial y},\\ \rho F_y &= -\frac{\partial}{\partial y}\frac{\mathbf{B}_x^2 + \mathbf{B}_y^2 + \mathbf{B}_z^2}{2} + \mathbf{B}_x\frac{\partial B_y}{\partial y} + \mathbf{B}_y\frac{\partial \mathbf{B}_y}{\partial x},\end{aligned}\right\} \tag{1.10}$$

where $\mathbf{B}_x$, $\mathbf{B}_y$ and $\mathbf{B}_z$ are the vector components of magnetic field intensity.

The magnetic field intensity is determined by the Maxwell equations, which in the case of a boundary layer may be written in the form:

$$\left.\begin{aligned}&\frac{\partial \mathbf{B}_x}{\partial x} + \frac{\partial \mathbf{B}_y}{\partial y} = 0,\\ &\frac{1}{\sigma_e}\frac{\partial^2 \mathbf{B}_x}{\partial y^2} = -\frac{\partial}{\partial y}(v_x\mathbf{B}_y - v_y\mathbf{B}_x),\end{aligned}\right\} \tag{1.11}$$

$$\left.\begin{aligned}&\frac{1}{\sigma_e}\frac{\partial^2 \mathbf{B}_y}{\partial y^2} = -\frac{\partial}{\partial x}(v_x\mathbf{B}_y - v_y\mathbf{B}_x),\\ &\frac{1}{\sigma_e}\frac{\partial^2 \mathbf{B}_z}{\partial y^2} = \frac{\partial v_x}{\partial x}\mathbf{B}_z + \frac{\partial v_y}{\partial y}\mathbf{B}_z.\end{aligned}\right\} \tag{1.12}$$

In the case of a conducting medium, $\rho\epsilon = (\mathbf{j}^2/\sigma)$ is the Joule heat, where $\mathbf{j}$ is the current density vector ($\mathbf{j} = \nabla \times \mathbf{B}$).

Equations (1.2) (1.4) and (1.5), the components of a friction tensor, diffusion flow vector and heat flow vector will be expressed differently in cases of laminar and turbulent boundary layers.

In the first case, if thermo-diffusion is to be neglected, then τ_{xy}, q_y and J_y^i may be determined from the following formulae:

$$\left.\begin{aligned}\tau_{xy} &= \mu\frac{\partial v_x}{\partial y},\\ J_y^i &= \rho D_i\frac{\partial \xi_i}{\partial y},\\ q_y &= \frac{\lambda}{c_p}\left\{\frac{\partial h}{\partial y} + \sum_i (Le_i - 1)h_i\frac{\partial \xi_i}{\partial y}\right\},\end{aligned}\right\} \tag{1.13}$$

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where

$$Le_i = \frac{\rho c_p D_i}{\lambda};\quad c_p = \sum_i \xi_i \frac{\partial h_i}{\partial T_i} = \sum_i \xi_i c_{pi};$$

$$h_i = \frac{u_{i0}}{M_i} + \int_0^T c_{pi}\,dT;\quad h = \Sigma\, \xi_i h_i.$$

In the case of a turbulent boundary layer, if we proceed from the semi-empirical theory of turbulence, it may be assumed [1]:

$$\left.\begin{aligned} \tau_{xy} &= \rho l^2 \left(\frac{\partial v_x}{\partial y}\right)^2, \\ J_y^i &= \rho l l_y^i \frac{\partial v_x}{\partial y}\frac{\partial \xi_i}{\partial y}, \\ q_y &= \rho l \frac{\partial v_x}{\partial y}\left[l_h \frac{\partial h}{\partial y} + \sum_i (l_d^i - l_h) h_i \frac{\partial \xi_i}{\partial y}\right], \end{aligned}\right\} \quad (1.14)$$

where l, l_d^i and l_h are mixing lengths for momentum, diffusion and heat transfer, respectively, which in boundary layer problems will be assumed to be proportional to the distance from the wall.

Moreover, considering a turbulent boundary layer, we shall assume that near the wall there is a laminar sublayer δ_l, wherein formula (1.13) is valid, the derivatives of heat content velocity and concentration undergoing discontinuity when passing through it, namely:

$$\left(\frac{\partial v_x}{\partial y}\right)_{y=\delta_l-0} = K_1 \left(\frac{\partial v_x}{\partial y}\right)_{y=\delta_l+0},$$

$$\left(\frac{\partial h}{\partial y}\right)_{y=\delta_l-0} = K_2 \left(\frac{\partial h}{\partial y}\right)_{y=\delta_l+0},$$

$$\left(\frac{\partial \xi_i}{\partial y}\right)_{y=\delta_l-0} = K_3 \left(\frac{\partial \xi_i}{\partial y}\right)_{y=\delta_l+0}.$$

If $l_d^i = l_h = l$, then the coefficients K_1, K_2 and K_3, as can easily be shown, are connected with the relations:

$$K_3 = K_1 \frac{\mu}{\rho D_i},\quad K_2 = K_1 \frac{c_p \mu}{\lambda}.$$

Diffusion, viscosity and heat conductivity coefficients for a gas mixture depend upon temperature as well as upon concentration of certain components. At present there are no complete theoretical and experimental data on their determination for multicomponent mixtures.

There is a number of formulae for binary mixtures. For example, in Wilke's work [38] formula is suggested for the calculation of two-gases mixture viscosity:

$$\mu = \frac{\mu_1}{1 + \xi_{12}(\chi_2/\chi_1)} + \frac{\mu_2}{1 + \xi_{21}(\chi_1/\chi_2)},$$

where

$$\xi_{ij} = \frac{[1 + (M_i/M_j)^{1/4}\sqrt{(\mu_i/\mu_j)}]^2}{2\sqrt{2}\sqrt{[1 + (M_i/M_j)]}}.$$

The dependence of viscosity upon temperature is well described by the Saterland formula [2]:

$$\mu = \mu(T_\infty)\frac{T_\infty + S_{\text{mix}}}{T + S_{\text{mix}}}\left(\frac{T}{T_\infty}\right)^{1.5},$$

where

$$S_{\text{mix}} = \frac{M_2}{2}\left(\frac{S_2}{M_2} + \frac{S_1}{M_1}\right)$$

is the Saterland constant. The dependence of heat conductivity upon temperature can be expressed through the number $Pr = (c_p\mu/\lambda)$ and the dependence $\mu(T)$.

The question of determining and calculating the viscosity coefficient, heat conductivity and diffusion for air has been discussed in detail [3, 4]. As far as the velocity of change of concentration on account of chemical reactions K_i is concerned, it may be represented in the form:

$$K_i = M_i \sum_{\alpha=1}^{r} \nu_{i\alpha} W_\alpha^i.$$

The equation of α reaction in chemical symbols is written in the form:

$$\sum_{i=1}^{q} \nu_{i\alpha} A_i = 0.$$

Since $\sum_{i=1}^{q} \mathbf{J}_i = 0$ from the determination of the diffusion vector, from equations (1.1) and (1.3) we have that $\sum_{i=1}^{q} \nu_{i\alpha} M_i = 0$.

The reaction rate W_α, according to the mass action law, will be written thus:

$$W_\alpha^i = K_\alpha^i c_1^{\nu_{1\alpha}} c_2^{\nu_{2\alpha}} \ldots - K_\alpha'^i c_1'^{\nu_{1\alpha}} c_2'^{\nu_{2\alpha}} \ldots,$$

where K_a^i and $K_a'^i$ are rate constants of a direct and inverse reaction, respectively; c_i, $c_2 \ldots$, c_1', c_2' are concentrations $[c_i = (\xi_i/M_i)\,\rho]$.

The reaction rate constants are functions of a gas mixture state. At quasi-equilibrium processes they may be considered as functions of temperature. In many cases they may be expressed according to the Arrhenius law in the form:

$$K_a^i = A_i(T)\,\mathrm{e}^{-E/RT},$$

where $A_i(T)$ is some constant or function of temperature.

The constant of a back-reaction is determined by means of the chemical equilibrium constants, K_p^{ai} and K_a^i, by the formula:

$$\frac{K_a'^i}{K_a^i} = K_p^{ai}(RT)^{-\sum_i \nu_{ia}}.$$

The question of determining reaction rate constants for dissociation of two-component mixtures and for air is discussed in [5, 6, 7, 8, etc.]. However, data on the determination of K_a^i and $K_a'^i$ in the case of dissociation and recombination are not available.

For the approximate solution of boundary layer equations it is assumed that there occurs a local chemical equilibrium. Then, concentrations, ξ_i, as functions of pressure and temperature, are determined on the basis of both the mass action law and the equations of conservation of atomic number. In this case thermodynamic functions of air have been determined and presented, for example, in tables given by Predvoditelev [9]. Equation (1.3) is excluded from consideration.

What may be the error of such an approximate solution has not yet been established.

2. TRANSFORMATION OF EQUATIONS TO THE DORODNITSIN VARIABLES

Let us introduce the values being as independent variables

$$\xi = \int_0^x f(x, u, T_w)\,\frac{p}{p_\infty}\,r^{2\epsilon}\,\mathrm{d}x,$$

$$\eta = \int_0^y \frac{\rho}{\rho_\infty}\,r^{\epsilon}\,\mathrm{d}y.$$

By introducing a new function instead of v_y:

$$V_y = \frac{\rho}{\rho_\infty}\,\frac{p_\infty}{p}\,\frac{r^{-\epsilon}}{f}\,v_y + r^{-2\epsilon}\frac{p_\infty}{p}\,\frac{u_x}{f}\,\frac{\partial \eta}{\partial x},$$

and turning to dimensionless variables $u = u_{\max}\,\bar{u}$

$$v_x = \bar{v}_x u_{\max}, \quad V_y = \bar{V}_y\,\frac{\nu_\infty}{L u_{\max}}\,u_{\max},$$

$$\xi = \bar{\xi}\,\frac{u_{\max}\,L^2}{\nu_\infty}, \quad \eta = \bar{\eta}L, \quad H = H_\infty(1+g),$$

$$h_i = H_\infty(1+g_i), \quad Z_i = \frac{\xi_i}{\xi_{i\delta}}$$

where

$$u_{\max}^2 = \frac{2H_\infty}{A};$$

L is the characteristic length having the order of thickness of a boundary layer; after certain transformations the equations determining the laminar motion of a gas mixture (ignoring radiation) may be written in the form (bars omitted in future):

$$\frac{\partial v_x}{\partial \xi} + \frac{\partial V_y}{\partial \eta} = 0, \tag{2.1}$$

$$v_x\frac{\partial v_x}{\partial \xi} + V_y\frac{\partial v_x}{\partial \eta} = \frac{\rho_0}{\rho}\,u\,\frac{\mathrm{d}u}{\mathrm{d}\xi} + \frac{\partial}{\partial \eta}\left(K\frac{\partial v_x}{\partial \eta}\right), \tag{2.2}$$

$$v_x\frac{\partial Z_i}{\partial \xi} + V_y\frac{\partial Z_i}{\partial \eta} = \frac{\partial}{\partial \eta}\left(\frac{K}{Sc_i}\,\frac{\partial Z_i}{\partial \eta}\right) + \bar{W}_i \tag{2.3}$$

$$\left.\begin{aligned} v_x\frac{\partial g}{\partial \xi} + V_y\frac{\partial g}{\partial \eta} &= \frac{\mu_\infty}{\rho_\infty}\,\frac{\partial}{\partial \eta}\,K\left[\frac{1}{Pr}\,\frac{\partial g}{\partial \eta}\right. \\ &+ \left(1 - \frac{1}{Pr}\right)\frac{\partial}{\partial \eta}\,\frac{v_x^2}{2} \\ &\left. + \sum_i \frac{Le_i}{Pr}\left(1 - \frac{1}{Le_i}\right)\xi_{i\delta}\,(1+\bar{g}_i)\,\frac{\partial Z_i}{\partial \eta}\right], \end{aligned}\right\} \tag{2.4}$$

where

$$K = \frac{\mu\rho p_\infty}{\mu_\infty \rho_\infty p}\,\frac{1}{f} = \frac{N}{f}, \quad Pr = \frac{c_p\mu}{\lambda},$$

$$\bar{W}_i = K_i\frac{p_\infty}{p\rho}\,\frac{r^{-2\epsilon}}{f}\,\frac{L^2}{\nu_\infty \xi_{i\delta}}, \quad \frac{1}{Sc_i} = \frac{Le_i}{Pr}.$$

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Let us represent boundary conditions for system, equations (2.1) to (2.4) as follows:

$$\left.\begin{aligned} &\eta = 0, \quad v_x = 0, \quad V_y = V_0, \quad Z_i = Z_{wi}, \\ &g = \frac{H_w}{H_\infty} - 1 = g_w, \quad \check{g}_i = \frac{h_{iw}}{H_\infty} - 1 \\ &\eta \to \infty \; v_x = u, \quad Z_i = 1, \quad g = 0. \end{aligned}\right\} \quad (2.5)$$

In the absence of volumetric heat release and mass forces, provided that H_∞ and $\xi_{i\delta}$ are assumed to be constant, integral relations in the Dorodnitsin variables will be written as follows:

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\delta^{**} + \frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}\xi}(2\delta^{**} + \delta^{*}) = \frac{K_w}{u}\nu_\infty\left(\frac{\partial}{\partial\eta}\frac{v_x}{u}\right)_{\eta=0} + V_w, \quad (2.6)$$

$$\frac{\mathrm{d}}{\mathrm{d}\xi}(\delta_g^* u) + uV_w\left(\frac{\xi_{iw}}{\xi_{i\delta}} - 1\right) = \frac{\rho_w D_{iw}}{\mu_w} K_w \nu_\infty \left(\frac{\partial Z_i}{\partial\eta}\right)_{\eta=0} + \int_0^\infty \frac{p_\infty}{p}\frac{W_i}{f}\frac{}{\rho r^{2\epsilon}}\,\mathrm{d}\eta, \quad (2.7)$$

$$\left.\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}\xi}(u\Delta^{**}) + uV_w\left(\frac{H_w}{H_\infty} - 1\right) \\ &= \frac{K_w}{Pr_w}\nu_\infty\left\{\left(\frac{\partial}{\partial\eta}\frac{H}{H_\infty}\right)_{\eta=0}\right. \\ &\left. + \sum_i \frac{\rho_w D_{iw}}{\mu_w} Pr_{iw}\left(1 - \frac{1}{Le_{iw}}\right)\frac{h_{iw}}{H_\infty}\frac{\partial\xi_i}{\partial\eta}\right|_{\eta=0}\Bigg\}. \end{aligned}\right\} \quad (2.8)$$

Here

$$\left.\begin{aligned} V_w &= \frac{\rho_w v_w}{\rho_\infty u}\frac{p_\infty}{p}\frac{r^{-\epsilon}}{f}, \\ \delta^{**} &= \int_0^{\delta_1}\frac{u_x}{u}\left(1 - \frac{v_x}{u}\right)\mathrm{d}\eta, \\ \delta^{*} &= \int_0^{\delta_1}\left(\frac{\rho_0}{\rho} - \frac{v_x}{u}\right)\mathrm{d}\eta, \\ \delta_g^{*} &= \int_0^{\delta_3}\frac{v_x}{u}(1 - Z_i)\,\mathrm{d}\eta, \\ \Delta^{**} &= \int_0^{\delta_3}\frac{v_x}{u}\left(1 - \frac{H}{H_\infty}\right)\mathrm{d}\eta \end{aligned}\right\} \quad (2.9)$$

where δ_1, δ_2 and δ_3 respectively denote thicknesses of dynamic, thermal and diffusion boundary layers in the Dorodnitsin variables. The thicknesses mentioned are determined from the conditions:

$$\left.\frac{\partial v_x}{\partial\eta}\right|_{\eta=\delta_1} = 0, \quad \left.\frac{\partial Z_i}{\partial\eta}\right|_{\eta=\delta_3} = 0, \quad \left.\frac{\partial H}{\partial\eta}\right|_{\eta=\delta_3} = 0.$$

[It is assumed in equation (2.8) that $(q_y + Av_x\tau_{xy})_{\eta=\delta_3} = 0$.] In the case of an asymptotic boundary layer in equation (2.3) it must be assumed that $\delta_1 = \delta_2 = \delta_3 = \infty$. In the case of equilibrium dissociation with the assumption that $(T/M) = ah + b$ the thickness of displacement δ^* is represented in the form:

$$\delta^* = \frac{1}{1 - A(u^2/2H_\infty) + (b/aH_\infty)} \times \left\{\frac{b}{aH_\infty}\delta_1^* + \Delta^* + A\frac{u^2}{2H_\infty}\delta^*\right\}$$

where

$$\delta_1^* = \int_0^{\delta_1}\left(1 - \frac{v_x}{u}\right)\mathrm{d}\eta, \quad \Delta^* = \int_0^{\delta_1}\left(\frac{H}{H_\infty} - \frac{v_x}{u}\right)\mathrm{d}\eta.$$

3. POSSIBLE SIMILARITY SOLUTIONS OF LAMINAR BOUNDARY LAYER EQUATIONS

Let us find out at what values of $u(\xi)$, $V_0(\xi)$, $g_w(\xi)$ and $Z_{iw}(\xi)$ solutions of equations (2.1) to (2.4) dependent on one variable $\tau = \eta x(\xi)$ are possible.

Assuming

$$\frac{v_x}{u} = \frac{\partial F(\tau, \xi)}{\partial\tau}$$

$$g = g_w\varphi(\tau, \xi),$$

$$Z_i = Z_i(\tau, \xi),$$

from equation (2.1) we shall have:

$$V_y = V_0 - \frac{u}{x}\frac{\partial F}{\partial\xi} - \frac{ux'}{x^2}\tau\frac{\partial F}{\partial\tau} + F\left(\frac{ux'}{x'} - \frac{u'}{u}\right) \quad (3.1)$$

and equations (2.2) to (2.4) will be reduced to the form:

$$\left.\begin{aligned}
&\left(\frac{\partial F}{\partial\tau}\frac{\partial^2 F}{\partial\tau\partial\xi}-\frac{\partial F}{\partial\xi}\frac{\partial^2 F}{\partial\tau^2}\right)\frac{c_6}{(\mathrm{d}/\mathrm{d}\xi)\ln(x/u)}\\
&+\left[\frac{V_0 f}{x}+c_6 F\right]\frac{\partial^2 F}{\partial\tau^2}\\
&=\frac{\mathrm{d}u/\mathrm{d}\xi}{u}\frac{c_6}{(\mathrm{d}/\mathrm{d}\xi)\ln(x/u)}\\
&\left[\frac{\rho_0}{\rho}-\left(\frac{\partial F}{\partial\tau}\right)^2\right]+\frac{\partial}{\partial\tau}N\frac{\partial F}{\partial\tau};
\end{aligned}\right\}\quad(3.2)$$

$$\left.\begin{aligned}
&\frac{c_6}{(\mathrm{d}/\mathrm{d}\xi)\ln(x/u)}\left\{\frac{\mathrm{d}\ln g_w}{\mathrm{d}\xi}\frac{\partial F}{\partial\tau}\varphi+\frac{\partial F}{\partial\tau}\frac{\partial\varphi}{\partial\xi}\right.\\
&\left.-\frac{\partial F}{\partial\xi}\frac{\partial\varphi}{\partial\tau}\right\}+\left[\frac{V_0 f}{x}+c_6 F\right]\frac{\partial\varphi}{\partial\tau}\\
&=\frac{\partial}{\partial\tau}N\left[\frac{1}{Pr}\frac{\partial\varphi}{\partial\tau}+\left(1-\frac{1}{Pr}\right)\frac{u^2}{g_w}\frac{\partial}{\partial\tau}\right.\\
&\times\left(\frac{\partial F}{\partial\tau}\right)^2+\sum_i\frac{Le_i}{Pr}\left(1-\frac{1}{Le_i}\right)\\
&\left.\times\frac{\xi_i\delta}{g_w}(1+\tilde{g}_i)\frac{\partial Z_i}{\partial\tau}\right];
\end{aligned}\right\}\quad(3.3)$$

$$\left.\begin{aligned}
&\frac{c_6}{(\mathrm{d}/\mathrm{d}\xi)\ln(x/u)}\left\{\frac{\mathrm{d}}{\mathrm{d}\xi}\ln\xi_{i\delta}Z_i\frac{\partial F}{\partial\tau}+\frac{\partial F}{\partial\tau}\frac{\partial Z_i}{\partial\xi}\right.\\
&\left.-\frac{\partial Z_i}{\partial\tau}\frac{\partial F}{\partial\xi}\right\}+\left[\frac{V_0 f}{x}+c_6 F\right]\frac{\partial Z_i}{\partial\tau}\\
&=\frac{\partial}{\partial\tau}\left(N\frac{Le_i}{Pr}\frac{\partial Z_i}{\partial\tau}\right)+\tilde{W}_i,
\end{aligned}\right\}\quad(3.4)$$

where

$$\tilde{W}_i=\bar{W}_i\frac{f}{x^2};\quad c_6=\frac{fu}{x^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\ln\frac{x}{u}.$$

As is seen from the system of equations given F, φ and Z_i will be functions of τ if the following conditions are fulfilled:

$$\left.\begin{aligned}
&N=N_1(\xi)N_2(\tau),\\
&\frac{V_0 f}{N_1 x}=c_3=\text{const},\\
&\frac{c_6}{N_1}=-c_1=\text{const},
\end{aligned}\right\}\quad(3.5)$$

$$\left.\begin{aligned}
&w_2(\xi)\frac{f(\mathrm{d}u/\mathrm{d}\xi)}{N_1x^2}=\text{const},\\
&w_4(\xi)\frac{f(\mathrm{d}u/\mathrm{d}\xi)}{N_1x^2}=\text{const},
\end{aligned}\right\}\quad(3.6)$$

where it is assumed that

$$\frac{\rho_0}{\rho}-\left(\frac{\partial F}{\partial\tau}\right)^2=w_1(\tau)w_2(\xi)+w_3(\tau)w_4(\xi),$$

$$\left(1-\frac{1}{Pr}\right)\frac{u^2}{g_w},\quad Pr,$$

$$\left(1-\frac{1}{Le_i}\right)\frac{(1+\tilde{g}_i)\xi_{i\delta}}{g_w}\quad\frac{Le_i}{Pr},\quad\tilde{W}_i$$

are functions of τ or constants;

$$\left.\begin{aligned}
&\frac{c_6}{N_1(\mathrm{d}/\mathrm{d}\xi)\ln(x/u)}\frac{\mathrm{d}\ln g_w}{\mathrm{d}\xi}\\
&\text{and}\\
&\frac{c_6}{N_1(\mathrm{d}/\mathrm{d}\xi)\ln(x/u)}\frac{\mathrm{d}\ln\xi_{i\delta}}{\mathrm{d}\xi}
\end{aligned}\right\}\quad(3.7)$$

are constant values. If the assumption that $(T/M)=ah+b$ is valid, then:

$$\left.\begin{aligned}
&w_1(\tau)=\varphi(\tau),\\
&w_2(\xi)=\frac{g_w}{1-u^2+(b/aH_\infty)},\\
&w_3(\tau)=1+\frac{b}{aH_\infty}\left[1-\left(\frac{\partial F}{\partial\tau}\right)^2\right]-\left(\frac{\partial F}{\partial\tau}\right)^2,\\
&w_4(\xi)=\frac{1}{1-u^2+(b/aH_\infty)}.
\end{aligned}\right\}\quad(3.8)$$

In the case of a plate or a cone, the conditions of equation (3.6) are always fulfilled and the remainder with the assumption that $f=N_1$ give the following equations for determining $x(\xi)$, $V_0(\xi)$, $g_w(\xi)$ and $\xi_{i\delta}(\xi)$:

$$\left.\begin{aligned}
&x(\xi)=\frac{1}{\sqrt{\{2[(c_1/u)\,\xi+c_2]\}}},\quad V_0=c_3x,\\
&\xi_{i\delta}=c_8\left(\xi+\frac{c_2u}{c_1}\right)^{c_8/c_7},\\
&g_w=c_5\left(\xi+\frac{c_2u}{c_1}\right)^{c_4/c_1},
\end{aligned}\right\}\quad(3.9)$$

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where c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8 are arbitrary constants. And

$$N_2, \quad Pr, \quad \frac{Le_i}{Pr}, \quad \left(1 - \frac{1}{Le_i}\right)(1 + \tilde{g}_i)\frac{\xi_{i\delta}}{g_w}$$

have to be functions of τ or constants.

Similar solutions of equations (3.2) to (3.4) are presented, for example, in the case of a plate for some particular cases [10–13]. Ordinary equations obtained in this case are simply solved when Pr and Le_i are constant, $N_2(\tau) = \text{const}$, $g_w = \text{const}$, $\xi_{i\delta}' = \text{const}$, $\tilde{W}_i = 0$, and injection is absent. In the presence of injection and variable numbers Pr and Le_i the equations are solved numerically. It will be noted that the solution of the ordinary equations obtained in the presence of chemical reactions, i.e. at $\tilde{W}_i \neq 0$, has not yet been actually considered (see, however, [12] and [14]). The case with variables g_w, $\xi_{i\delta}$ has not been considered either.

In the case of a wing or axisymmetrical body the conditions that $[1 - (1/Pr)]\,u^2/g_w$ is the function of τ or the constant (since u is the function of ξ) may be fulfilled only when:

(a) $Pr = 1$

(b) $\dfrac{u^2}{g_w} = \text{const},$

and Pr is the function of τ or the constant.

In the case of hypersonic velocities, when a boundary layer of thin bodies is considered, it may be assumed that $u^2 \approx 1$, then the condition $u^2/g_w = \text{const}$ will be fulfilled at $g_w = \text{const}$. When a boundary layer near a stagnation point is considered, u^2 may be assumed approximately equal to zero. The conditions of equation (3.6) are fulfilled not at each dependence ρ_0/ρ upon T. In particular, for their fulfillment it is necessary that $w_2(\xi)/w_4(\xi) = \text{const}$, which with the assumption that the equation $(T/M) = ah + b$ is valid leads to the condition $g_w = \text{const}$.

In this case equations (3.5) and (3.6) lead to the following equations for determining $x(\xi)$, $u(\xi)$ and $V_0(\xi)$ where there are similar solutions:

$$f V_0 = c_3 N_1 x, \quad \frac{fu}{N_1 x^2}\left(\frac{x'}{x} - \frac{u'}{u}\right) = -c_1$$

$$\frac{f(\mathrm{d}u/\mathrm{d}\xi)}{N_1 x^2[1 - u^2 + (b/aH_\infty)]} = \beta. \tag{3.10}$$

From the latter equations we get:

$$x = \sqrt{\left(\frac{f(\mathrm{d}u/\mathrm{d}\xi)}{\beta N_1[1 - u^2 + (b/aH_\infty)]}\right)} = \sqrt{\left(\frac{(p_\infty/p)\,r^{-2\varepsilon}\,(\mathrm{d}u/\mathrm{d}x)}{\beta N_1[1 - u^2 + (b/aH_\infty)]}\right)} \tag{3.11}$$

and for $u(\xi)$, the following equation:

$$\frac{f}{N_1}\,\frac{\mathrm{d}u/\mathrm{d}\xi}{\beta[1 - u^2 + (b/aH_\infty)]} = D^2\,\frac{u^{-(1/m)+2}}{[1 - u^2 + (b/aH_\infty)]^{-1/2m}}$$

or

$$\frac{(p_\infty/p)\,r^{-2\varepsilon}\,(\mathrm{d}u/\mathrm{d}x)}{N_1\beta[1 - u^2 + (b/aH_\infty)]} = D^2\,\frac{u^{-(1/m)+2}}{[1 - u^2 + (b/aH_\infty)]^{-1/2m}},$$

where

$$\frac{1}{2m} = \frac{c_1}{\beta[1 + (b/aH_\infty)]}.$$

The term entering equation (3.2)

$$\frac{1}{u}\,\frac{\mathrm{d}u}{\mathrm{d}\xi}\,\frac{c_6}{(\mathrm{d}/\mathrm{d}\xi)\ln(x/u)}\left[\frac{\rho_0}{\rho} - \left(\frac{\partial F}{\partial \tau}\right)^2\right] = \beta\left[1 + g_w q - \left(\frac{\partial F}{\partial \tau}\right)^2\left(1 + \frac{b}{aH_\infty}\right) + \frac{b}{aH_\infty}\right].$$

The form of the function $u(x)$ depends on parameters β, m and D, and that of $u(\xi)$ on the form of the function $f(\xi)$ as well.

In particular, assuming that

$$f = N_1\left(1 - u^2 + \frac{b}{aH_\infty}\right)^{1+(1/2m)},$$

we shall get $u(\xi)$ in the form:

$$u = c_{11}(\xi + c_{12})^n \tag{3.13}$$

if $m \neq 1$, and

$$u = c_{11}\,e^{D^2\beta\xi} \tag{3.14}$$

if $m = 1$, where $n = [m/(1 - m)]$,

$$x(\xi) = D\,\frac{u^{-(1/2m)+1}}{\{\sqrt{[1 - u^2 + (b/aH_\infty)]}\}^{-1/2m}},$$

c_{11}, c_{12} are arbitrary constants.

If we assume $f = uN_1$ we shall obtain

$$x = u\sqrt{\left(\frac{1}{2(c_1\xi + D)}\right)} \tag{3.15}$$

and

$$u = c\,\frac{[\xi + (D/c_1)]^m\sqrt{[1 + (b/aH_\infty)]}}{\sqrt{\{1 + c^2[\xi + (D/c_1)]^{2m}\}}}.$$

The variables ξ and τ turning into the Mangler variables.

The numerical solutions of equations (3.2) and (3.3) under boundary conditions:

$$F(0) = F'(0) = 0; \quad F'(\infty) = 1;$$

$$g(0) = 1; \qquad g(\infty) = 0.$$

at $b = 0$, $N_1 = 1$, $Pr = 1$, $Le_i = 1$ and $m \neq 1$ over a wide range of β (up to $\beta = 2$) were obtained by a number of investigators [15–17]. Moreover, Cohen and Reshotko [17] obtained a numerical solution of the mentioned equations at $Pr = 0{\cdot}7$ and $(2/g_w) = 1{\cdot}2$.

Similar solutions may also be used in studying a motion near the critical point where $u \approx x(\mathrm{d}u/\mathrm{d}x)_0$. It is convenient to introduce here the Mangler variables. Numerical solutions of equations for this case are presented by Reshotko and Cohen [18] for $Le_i = 1$, $N_1 = 1$ and by Li and Nagamatsi [16] and by Lees [19] for $Pr = 1$ and $Pr = 0{\cdot}7$. In the neighbourhood of the critical point at $Le_i \neq 1$ for final reaction rates for a catalytic and non-catalytic surface, the numerical solution of equations (3.2) to (3.4) was obtained by Fay, Riddell and Kemp [20, 21]. The solution of these equations for injection and also for another change law of $u(\xi)$ has not been discussed in detail. Similar solutions for chemical reaction and surface melting also demand a more detailed analysis.

4. APPROXIMATE METHODS FOR SOLVING LAMINAR BOUNDARY LAYER EQUATIONS

As was mentioned above, similar solutions may occur both with definite velocity distribution of external flow along the edge of the boundary layer and for certain ranges of the numbers Pr, Le_i, g_w, etc. In the general case of arbitrary pressure distribution along a profile and with arbitrary boundary conditions the problem has to be solved approximately.

In the boundary-layer theory the following arbitrary methods are applied:

(1) A partial application of local similarity in which the pre-history of a flow is taken into consideration only as the dependence of coefficients on the variable ξ.

(2) Methods of integral relations.

(3) Methods involving expansion in series in which the coefficients of each variable are obtained by solving ordinary differential equations, and the expansion variables depend on the conditions outside the boundary layer.

(4) Methods of successive approximation.

(5) Difference methods for solving laminar boundary-layer equations.

(6) Methods of small disturbances.

We shall now consider on these methods:

(1) *Local similar solutions*

It is the essence of this method that the problem is reduced to solving systems (3.2) and (3.4) with local values β, g_w, (u^2/H_∞) and N determined at each point of ξ.

Such a method is applicable if it is known beforehand that the external flow and gas properties on a body depend little on ξ, i.e. when derivatives with respect to ξ are small compared with those with respect to η.

(2) *Method of integral relations*

The application of this method to solving the problem of gas motion in a laminar boundary layer is different both for a finite thickness layer and a symptotic one. For the layer of finite thickness it is assumed that the profile of velocities, heat contents and concentration may be represented as polynomials of ratios $(1/\delta_i)$, where δ_i is the corresponding thickness, the coefficients of which are determined from the conditions on the wall and on the edge of a boundary layer. From the integral relations we get ordinary differential equations for determining boundary layer thicknesses. The wall conditions are obtained from the differential equations, assuming their validity on the wall,

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their number being able to increase by differentiating the equations. For a thermoisolated profile the method in question was used in a number of works [22–24]. The calculations become rather complicated under more general wall conditions.

For the asymptotic boundary layer, the integral relation method is used, with corresponding profiles taken from a similar solution. In the foreign literature such a conception was first used by Thwaites [25], who considered a definite relation in which tangential wall stress, local pressure gradient and the ratio of displacement thickness to the impulse loss thickness are interrelated. The Thwaites method was worked out by Rott and Crabtrec [26] who considered the case for $Pr = 1$ and a thermoisolated surface, and also by Cohen and Reshotko [17] for $Pr = 1$, g_w is constant. In Cohen and Reshotko's version the enthalpy gradient on the wall, $g_w\varphi'(0)$, is assumed to be, as a result of similar solution, the known function β of the parameter connected by the definite relation with the pressure gradient and impulse loss thickness.

Later on this method was worked out by Hayes [27] who, in contrast to Thwaites, introduced the boundary layer thickness as the function $G(x)$ instead of the impulse loss thickness to separate the variables.

Methods utilizing local similar solutions are similar to the Kochin–Loitsyansky method [28] worked out for an incompressible liquid and generalized for a compressible one.

We shall now show how it is possible to solve a problem on a gas mixture motion in a laminar boundary layer by the method mentioned above.

If we assume that $f_1 = N_w$ and introduce the values $\bar{\delta}^{**} = \delta^{**}x$, $\bar{\delta}^*_g = x\delta^*_g$, $\bar{\Delta}^{**} = -(x/g_w)\Delta^{**}$ and substitute $(v_x/u) = (\partial F/\partial\tau)$ etc., the integral equations (2.6) to (2.8) may be written as follows:

$$\left.\begin{aligned} &u\delta^{**}\frac{\mathrm{d}\delta^{**}}{\mathrm{d}\xi} + \beta(\delta^{**})^2\left\{\left(1+\frac{b}{aH_\infty}\right)\frac{\delta_1^*}{\delta^{**}}\right.\\ &\left.+ g_w\frac{\Delta^*}{\delta^{**}} + \left[2\left(1+\frac{b}{aH_\infty}\right) - u^2\right]\right\}\\ &= \bar{\delta}^{**}F'''_{\tau\tau}(0,\xi) + \beta_1; \end{aligned}\right\} \quad (4.1)$$

$$u\delta^*_g\left(\frac{\mathrm{d}\delta^*_g}{\mathrm{d}\xi} + \delta^*_g\frac{\mathrm{d}\ln u}{\mathrm{d}\xi}\right) = \beta_1\bar{\delta}^*_g(1 - Z_{iw}) + \left(\frac{Le_i}{Pr}\right)_w\bar{\delta}^*_gZ'_{i\tau}(0,\xi); \quad (4.2)$$

$$\left.\begin{aligned} &u\Delta^{**}\left(\frac{\mathrm{d}\Delta^{**}}{\mathrm{d}\xi} + \Delta^{**}\frac{\mathrm{d}\ln u}{\mathrm{d}\xi}\right) - \beta_1 g_w^2\Delta^{**}\\ &= \frac{g_w}{Pr_w}\left\{g_w\varphi'_\tau(0,\xi)\Delta^{**}\right.\\ &\left.+\sum_i(Le_i - 1)\frac{h_{iw}\xi_{i\delta}}{H_\infty}\Delta^{**}z'_\tau(0,\xi)\right\} \end{aligned}\right\} \quad (4.3)$$

where

$$\beta = \frac{\delta^{**2}}{\bar{\delta}^{**2}}\frac{\mathrm{d}u/\mathrm{d}\xi}{1 - u^2 + (b/aH_\infty)}, \qquad \beta_1 = \frac{V_0\delta^{**}}{\bar{\delta}^{**}} = \frac{V_0}{x}. \quad (4.4)$$

For the approximate solution of the problem of gas motion in a boundary layer of the wing, when there are no similar solutions, let us assume that:

$$\frac{v_x}{u} = \frac{\partial F}{\partial\tau}, \quad \varphi = \varphi(\tau_3), \quad Z_i = Z_i(\tau_2),$$

where

$$\tau_1 = \frac{\eta}{\delta^{**}}\bar{\delta}^{**}, \quad \tau_2 = \frac{\eta}{\delta^*_g}\bar{\delta}^*_g, \quad \tau_3 = \frac{\eta}{\Delta^{**}}\bar{\Delta}^*_g$$

and the form of the functions $F(\tau_1)$, $\varphi(\tau_3)$ and $Z_i(\tau_2)$ are the same as in the case of similar solutions. In this case we shall consider the parameters β, β_1, g_w, Z_{iw} as functions of ξ; β, β_1 being expressed through δ^{**} and $u(\xi)$ from equations (4.4) where:

$$\bar{\delta}^{**} = \int_0^\infty\frac{\partial F}{\partial\tau}\left(1 - \frac{\partial F}{\partial\tau}\right)\mathrm{d}\tau,$$

$$\bar{\delta}^*_g = \int_0^\infty\frac{\partial F}{\partial\tau}(1 - Z_i)\,\mathrm{d}\tau,$$

and so on.

The values entering the integral equations $\bar{\delta}^{**}F_{\tau\tau}(0,\xi)$, δ_1^*/δ^{**} as functions of β and β_1; $\bar{\Delta}^*/\delta^{**}$, $\varphi'_\tau(0,\xi)\bar{\Delta}^{**}$ as functions of β, β_1, g_w, Le_i and Pr; δ^*_g/δ^{**}, $\bar{\delta}^*_gZ'_{i\tau}(0,\xi)$ as functions of β, β_1;

Z_{iw}, Le_i/Pr and Δ^{**}/δ^{**} as a function of β, β_1, g_w, Z_{iw}, Le_i and Pr are determined from similar solutions.

In solving the problem, δ^{**} is first determined as a function of ξ at given $u(\xi)$ and V_0 from equation (4.1) then $\beta(\xi)$ and $\beta_1(\xi)$, and lastly δ_g^* are determined. Friction, heat transfer and the diffusion flow are determined from the relations:

$$\tau_w = \mu_w \frac{\rho_w}{\rho_\infty} r^\epsilon u F''_{\tau\tau}(0, \xi) \frac{\delta^{**}}{\delta^{**}},$$

$$J_w^i = -\frac{\rho_w^2}{\rho_\infty} D_{iw} r^\epsilon \xi_{i\delta} Z'_{i\tau}(0, \xi) \frac{\delta_g^*}{\delta_g^*},$$

$$g_w = \mu_w \frac{\rho_w}{\rho_\infty} r^\epsilon \left\{ \frac{g_w}{H_\infty} \frac{\varphi'_\tau(0, \xi)}{\Delta^{**}} \Delta^{**} \frac{1}{Pr_w} + \sum_i \frac{Le_i - 1}{Pr_w} h_{iw} \xi_{i\delta} \frac{Z'_{i\tau}(0, \xi)\delta_g^*}{\delta_g^*} \right\}.$$

Equation (4.1) for the incompressible liquid was solved by Kochin and Loitsyansky. The solution of the latter equations for $Le_i = 1$ at $g_w = 0$ has been obtained [29].

The works devoted to the consideration of more general cases, as far as we know, are not available.

(3) *Expansion methods*

For simplicity we shall consider thermodynamic equilibrium at $Le_i = 1$. To apply the expansion method it is necessary to express all the parameters characterizing gas properties, (e.g. the density ratio ρ_0/ρ) analytically.

The velocity outside a boundary layer is represented as follows

$$u = u_0\xi^{1/4} + u_1\xi^{3/4} + u_2\xi^{5/4} \ldots = \sum_{K=0}^{\infty} u_K \xi^{(2K+1)/4}, \tag{4.5}$$

where u_K is a constant, and is determined from the non-viscous velocity distribution.

The unknown functions of F and g^* are also given as power series of ξ.

$$F = F_s + \sum_{K=1}^{\infty} F_K \xi^{K/2}, \quad g = g_s + \sum_{K=1}^{\infty} g_K \xi^{K/2}, \tag{4.6}$$

where F_K and g_K are considered to be dependent only on τ.

The first term in equation (4.5) is a non-viscous flow in the neighbourhood of the axisymmetrical critical point. Such a flow has been examined before, so that F_s and g_s are known. Substituting equation (9.6) into the equations and comparing coefficients at equal powers of ξ, we get ordinary differential equations for determining F_K and g_K. In the general case this method leads to very laborious calculations. Nevertheless, it can be simple enough if the non-viscous velocity distribution is represented by two or three terms of series of equation (4.5).

This method was used by Görtler [30] for a two-dimensional incompressible boundary layer without heat transfer; and by Sparrow [31] for the same problem with heat transfer. It can also be generalized for the case $Le_i \sim 1$, as well as for cases where diffusion is taken into account.

(4) *Methods of successive approximations*

This method applied to an incompressible liquid is given in Shvets' work [32]. The essence of this method is as follows. In the system, equations (1.2) to (1.4), for a first approximation inertial terms are neglected, and the solution of a simplified system with corresponding boundary conditions is found. The next approximation is obtained by substituting the first one into the inertial terms, and solving the system obtained for zero boundary conditions. The sum of this solution, and the first approximation gives the second approximation and so on. The thicknesses of dynamic, thermal and diffusion boundary layers are determined from the conditions:

$$\left(\frac{\partial v_x}{\partial y}\right)_{y=\delta_1} = 0, \quad \left(\frac{\partial h}{\partial y}\right)_{y=\delta_2} = 0, \quad \left(\frac{\partial \xi_i}{\partial y}\right)_{y=\delta_3} = 0.$$

Applying this method to a compressible liquid, it is convenient to introduce the variable $\bar{\eta} = \int_0^y (\mu/\mu_\infty)\, dy$, instead of the variable η.

Using this method Galanova [33] solved the problem for a laminar boundary layer of a plate in the presence of dissociation. The solution of the problem of a laminar boundary layer both of a plate and a wing in the presence of injection predetermined by the arbitrary law is given in

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work [34]. A good agreement of the results obtained by this method with the available exact solutions should be noted. The laborious nature of the calculations is a drawback of this method.

(5) *Difference methods*

Methods for the solution of laminar boundary layer equations employing assumptions (similarity, predetermined type of a velocity profile and enthalpy, velocity expansion outside the boundary layer by ξ with coefficients independent of ξ, etc.) which facilitated the problem of reducing a non-linear system in partial derivatives to one or several non-linear ordinary equations, were considered in previous sections. But even with such simplifications one has to resort to a numerical solution of the equations obtained. It is therefore of interest to consider methods for directly solving the partial differential equations without applying assumptions of similarity, etc.

For simplicity let us consider the case of thermodynamic equilibrium at $Le_i = 1$ [35].

Turning to Crocco's variables:

$$\bar{v}_x = \frac{v_x}{u}, \quad \xi = \int_0^x r^{2\epsilon}\, \rho_0 u\, \mathrm{d}x$$

and introducing the unknown functions:

$$\bar{\tau} = \frac{\mu\,(\partial v_x/\partial y)\,\sqrt{2\xi}}{r^{\epsilon}\rho_0\mu_0 u^2}, \quad H' = \frac{H - H_w}{H_\infty - H_w},$$

we shall write equations for determining $\bar{\tau}$ and H' as follows:

$$\left.\begin{aligned}
&A_1 \frac{\partial^2\tau}{\partial\bar{v}_x^2} + B_1 \frac{\partial\bar{\tau}}{\partial\xi} + C_1 \frac{\partial H'}{\partial\xi} + D_1 \frac{\partial\tau}{\partial\bar{v}_x} \\
&+ E_1 \frac{\partial H'}{\partial\bar{v}_x} + F_1 = 0, \\
&A_2 \frac{\partial^2 H'}{\partial\bar{v}_x^2} + B_2 \frac{\partial H'}{\partial\xi} + C_2 \frac{\partial\bar{\tau}}{\partial\xi} + D_2 \frac{\partial H'}{\partial\bar{v}_x} \\
&+ E_2 \frac{\partial\bar{\tau}}{\partial\bar{v}_x} + G_2 \left(\frac{\partial\bar{\tau}}{\partial\bar{v}_x}\right)\left(\frac{\partial H'}{\partial\bar{v}_x}\right) + F_2 = 0,
\end{aligned}\right\} \quad (4.7)$$

where the coefficients A_1, B_1, C_1, D_1, E_1, F_1, A_2, B_2, C_2, D_2, E_2, F_2 and G_2 are complex functions of all the initial values and also of ξ, $\bar{v}_x$, $\bar{\tau}$, h, Pr, $\mathrm{d}u/\mathrm{d}\xi$ and $K = (c_p/c_v)$.

Boundary conditions will take the form* at

$$\bar{v}_x = 0, \quad \frac{\partial\tau}{\partial\bar{v}_x} = f_1\left(\xi, u, \frac{p_0}{p_s}, \frac{\mathrm{d}u}{\mathrm{d}\xi}, \frac{h_0}{h_s}\right)$$
$$\bar{v}_x = 1, \quad \tau = 0, \quad H' = 0. \quad (4.8)$$

It can be shown that the system presented is of a parabolic type, and the characteristics are the lines $\xi = \text{const}$. It makes it possible to apply approximate methods using the connexion between the values of the function and those of its derivatives at three neighbouring points to the system presented. As a result the problem is reduced either to the solution of n algebraic equations with n unknowns, or to the system of the ordinary differential equations on each line $\xi = \text{const}$. There arise difficulties in obtaining data on the singular line.

The difference approximation of derivatives has been represented by ξ as well as by $\bar{v}_x$ in the form [35]:

$$\left.\begin{aligned}
&\frac{\partial H'}{\partial\xi} = \frac{H'_l - H'_\delta}{\Delta\xi}, \\
&\frac{\partial H'}{\partial\bar{v}_x} = \frac{1}{4\Delta\bar{v}_x}(H'_a - H'_c + H'_d - H'_f), \\
&\frac{\partial^2 H'}{\partial\bar{v}_x^2} = \frac{1}{2\Delta\bar{v}_x^2}(H'_a - 2H'_b + H'_c + H'_d \\
&\qquad - 2H'_e + H'_f), \\
&\left(\frac{\partial H'}{\partial\bar{v}_x}\right)\left(\frac{\partial\bar{\tau}}{\partial\bar{v}_x}\right) \\
&\quad = \frac{1}{8\Delta\bar{v}_x^2}[(H'_a - H'_i)(\bar{\tau}_d - \bar{\tau}_f) \\
&\quad + (H'_d - H'_f)(\bar{\tau}_a - \bar{\tau}_c)]
\end{aligned}\right\} \quad (4.9)$$

where symbols d, e, f, a, b, c refer to the series of three points chosen on the successive lines $\xi = \text{const}$.

On each line $\xi = \text{const}$ the system is reduced to $2(i-1)$ linear equations with $2(i-1)$ unknowns, H'_i and τ_i, of the form:

$$A_{1i}H'_{i+1} + B_{1i}H'_i + c_{1i}H'_{i+1} + D_{1i}\tau_{i+1} + E_{1i}\tau_i + F_{1i}\tau_{i-1} = G_{i1}.$$

* p_s is the pressure in the vicinity of a singular point

Symbol i is the index of the point on the predetermined series $\xi = \text{const}$ of such a point that $i = 0$ corresponds to $\bar{v}_x = 0$, $i = 1$, $\bar{v}_x = 1$.

Solving the system with respect to H_i and τ_i on a new line, using the conditions $\bar{v}_x = 0$ and $\bar{v}_x = 1$, we get the values of H'_i and $\bar{\tau}_i$ which are then used for determining these values on the next lines.

Calculations are made in two opposite directions: one, starting with the body boundary with fixed conditions at $\bar{v}_x = 0$ and the second, vice versa, from the boundary layer edge with fixed conditions at $\bar{v}_x = 1$. The conditions on the singular line $\xi = 0$ are obtained from equation (4.6). Assuming the existence and the only possible way of solving system, equation (4.7), with boundary conditions, equation (4.8), the solution being regular in the region $0 < \bar{v}_x < 1$ to $\epsilon < \xi < \infty$ one may come to the conclusion that:

$$\frac{\partial H'}{\partial \xi} = \frac{\partial \bar{\tau}}{\partial \xi} = 0 \text{ at } \xi = 0.$$

Because of the use of power dependences for air properties in [35] finite limits of coefficients at $\partial H'/\partial\xi$ and at $\partial\tau/\partial\xi$ in equation (4.7) were found.

Thus, to determine H', $\bar{\tau}$ on $\xi = 0$ we get a system of ordinary differential equations with two point boundary conditions, the solution of which is found by the Runge–Kudt–Hill method. The method mentioned above was used for calculating both a perfect dissociating gas and pressure distribution according to the Newton modified theory. $(p_0/p_s) = Sm^2\theta$.

Difference approximation, equation (4.8), may be replaced by the relations establishing the connexion between the values of the function and those of its derivatives at three adjacent points:

$$\xi = \xi_i; \quad \xi = \xi_{i+1} = \xi_i + h; \quad \xi = \xi_{i+2} = \xi_i + 2h \ldots;$$

where h is the step by ξ.

$$F(\xi_{i+2}, \bar{v}_x) = F(\xi_i \bar{v}_x) + 2h \left.\frac{\partial F}{\partial \xi}\right|_{\xi=\xi_{i+1}} + 0(h^3); \qquad (4.10)$$

$$\frac{\partial^n F(\xi_{i+2}, \bar{v}_x)}{\partial \bar{v}_x^n} = \frac{\partial^n F(\xi_i, \bar{v}_x)}{\partial \bar{v}_x^n} + 2h \left.\frac{\partial^{n+1} F(\xi, \bar{v}_x)}{\partial \xi \partial \bar{v}_x^n}\right|_{\xi=\xi_{i+1}} + 0(h^3); \qquad (4.11)$$

$$\left.\frac{\partial F}{\partial \xi}\right|_{\xi=\xi_{i+1}} = \frac{2}{h} F(\xi_{i+2}, \bar{v}_x) - \left[\frac{4}{h} F(\xi_{i+1}, \bar{v}_x) - \frac{2}{h} F(\xi_i \bar{v}_x) - \left.\frac{\partial F(\xi_1 \bar{v}_x)}{\partial \xi}\right|_{\xi=\xi_i}\right] + 0(h^2).$$

We understand the function F to be one of the functions H' or $\bar{\tau}$.

From equations (4.7), with the help of equation (4.11), we may eliminate the derivatives of H' and $\bar{\tau}$ with respect to ξ at $\xi = \xi_{i+2}$; and with the help of equation (4.10) in the coefficients A_2, $B_2 \ldots$ we may substitute all the values at $\xi_{i+2} = \xi$ for the values at the points $\xi = \xi_i$ and $\xi = \xi_{i+1}$. Thus, for the line $\xi = \xi_{i+2}$ we get a system of ordinary differential equations differing from exact system of equation (4.7) in values of the order h^2.

The derivatives

$$\frac{\partial H'}{\partial \xi}, \quad \left.\frac{\partial \tau}{\partial \xi}\right|_{\xi=\xi_{i+2}},$$

necessary for determining the approximate values of the functions, and their first derivatives at $\xi = \xi_{i+3}$ are determined from equation (4.7). Boundary conditions are linearized in an analogous way.

It is clear that for the boundary layer calculation by successive transition from the straight line $\xi = \xi_i$ to that $\xi = \xi_{i+1}$ while increasing ξ, one must know the values:

$$F(0, \bar{v}_x), \quad \frac{\partial F(0, \bar{v}_x)}{\partial \bar{v}_x}, \quad \left.\frac{\partial F}{\partial \xi}\right|_{\xi=\xi_1=h} \quad \left.\frac{\partial^2 F}{\partial \bar{v}_x^2}\right|_{\xi=\xi_1=h},$$

i.e. it is necessary to solve equation (4.7) at the points $\xi_0 = 0$, $\xi_1 = h$ in the neighbourhood of a front edge. At sufficiently small ξ, such a solution may be found with the help of expansion of H' and $\bar{\tau}$ in powers of ξ in the neighbourhood of the front edge:

$$H' = H'_{(0)}(\bar{v}_x) + H'_{(1)}(\bar{v}_x)\xi^{1/2} + H'_{(2)}(\bar{v}_x)\,\xi + \ldots$$
$$\bar{\tau} = \bar{\tau}_{(0)}(\bar{v}_x) + \bar{\tau}_{(1)}(\bar{v}_x)\,\xi^{1/2} + \bar{\tau}_{(2)}(\bar{v}_x)\,\xi + \ldots \qquad (4.12)$$

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Substituting the series of equation (4.12) into equation (4.7) and equating coefficients at equal powers of ξ to zero, we shall obtain a sequence of systems of ordinary differential equations for determining coefficients of series:

$$H'_{(0)}, \quad H_{(1)}, \quad \bar{\tau}_{(0)}, \quad \bar{\tau}_{(1)}, \ldots$$

The equations for $H'^{(0)}$ and $\bar{\tau}^{(0)}$ are non-linear, while the rest are linear ones. The derivatives of the functions with respect to ξ at $\xi = h$ are determined from equation (4.7) on the basis of the known values of the functions and their derivatives with respect to $\bar{v}_x$ at the point $\xi_1 = h$, calculated with the help of the series in equation (4.12).

It is true that in this case there is disagreement in the values of the functions and their derivatives with respect to $\bar{v}_x$ calculated from equation (4.12), near the surface $\bar{v}_x = 0$; therefore direct determination of the derivatives by ξ at $\xi_1 = h$ and $v_x = 0$ proves to be impossible. The values of the functions and their derivatives can be corrected near $\xi = 0$ using the series, equation (4.12), the approximations, equation (4.11), and the system, equation (4.7). This method, as well as all the methods described in section 4, may be applied to more complicated problems.

This method was applied by Kulonen [36] to solving the problem of the laminar boundary layer of a wing in the presence of injection.

(6) *Application of small disturbance method to laminar boundary layer problems*

If the velocity outside the boundary layer and the air properties dependent on temperature may be represented as small parameters which may be expanded with respect to $\epsilon_i (\epsilon_i^c \sim 0)$, then it is possible in many cases to solve laminar boundary layer problems by the method of small disturbances. The essence of the method lies in disturbance of the known solutions of boundary layer equations, expansion being done by the definite parameter.

This method has been applied to laminar boundary layer problems in the case of thermodynamic equilibrium, with an impenetrable wall and constant temperature distribution along the surface [37]. This method may also be applied to more complicated cases.

5. TURBULENT BOUNDARY LAYER

When solving turbulent boundary layer problems for a gas mixture one may generalize the semi-empirical theory of turbulence. Such a generalization is given in our work [1]. In it there is a complete solution of the problem on the turbulent boundary layer of a plate assuming the equality of mixing lengths $l = l_d = l_h = Ky$, where y is the distance from the wall.

This allows one to write integrals of energy and diffusion equations for the turbulent layer:

$$H = av_x + b, \quad \xi_i = a_1 v_x + b_1.$$

In the laminar sub-layer at Le_i and Pr numbers not equal to unity, H is assumed to be represented as the polynomial of the second power of v_x. When solving the problem, friction stress is taken to be:

$$\tau = \tau_w \left[1 - \left(\frac{y}{\delta}\right)^2\right] + \delta \frac{\mathrm{d}p}{\mathrm{d}x}\left[\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2\right],$$

where it is assumed that $T/M = c_1 h + d$, and, as usual, integral relations are used.

For determining the heat flow and the friction coefficients the following formulae were obtained in the work:

$$q_w = \frac{\lambda_w}{c_p}\frac{\tau_w}{\mu_w}\left\{Pr^{1/3}\frac{H_\infty - H_w - Aa(Pr)\,(u_i^2/2)}{u[1 - a(Pr)\,(u_l/u)]} - Pr\frac{f_1\,(u_i^2/2) - Pr\,f_2 u_l}{u[1 - a(Pr)\,(u_l/u)]} - Pr\,f_2\right\},$$

$$\tau_w = \tfrac{1}{2} c_f \rho_0 u^2, \quad c_f = \frac{1 - \bar{u}^2}{\bar{H}_w}\frac{2}{\xi^2}$$

$$n_1 + n_2 \frac{c}{\bar{u}_w} K\xi = \ln\left\{\frac{K^2}{D}\frac{c^2}{\bar{u}_w}\sqrt{\left(\frac{1 - \bar{u}^2}{\bar{H}_w}\right)\frac{ux}{\nu_w}}\right\},$$

where

$$a(Pr) = 1 - \sqrt{Pr}; \quad u_l = \frac{K_1}{K}\sqrt{\left(\frac{\tau_w}{\rho_w}\right)};$$

$$f_1 = Pr\,\mu_w \frac{\left\{\dfrac{\partial}{\partial y} - \dfrac{\mu}{Pr}\displaystyle\sum_i\left[(Le_i - 1)h_i\frac{\partial \xi_i}{\partial y}\right]\right\}_{y=0}}{\tau_w^2};$$

$$f_2 = f_1\frac{u_l}{Pr} + \frac{\lambda}{c_p \tau_w}\left(\sum_i (Le_i - 1)h_i\frac{\partial \xi_i}{\partial y}\right)_{y=\delta l}$$

$$\frac{K_1}{K} \simeq 11; \quad K \approx 0{\cdot}4; \quad n_1 \simeq 2{\cdot}9; \quad n_2 \approx 1{\cdot}16;$$

$$c = \arcsin\frac{2\bar{u}^2 + \bar{H}_w - 1}{\beta} - \arcsin\frac{\bar{H}_w - 1}{\beta};$$

$$D = \frac{K_1}{2K^2}\exp\left\{1 - K_1 - K_1\frac{a_1}{\beta^2}\{(\bar{H}_w + 1) - 2\sqrt{[(1-\bar{u}^2)\bar{H}_w]}\}\right\};$$

$$a_1 = (1 - \bar{H}_w)(1 - Pr) + \frac{Pr\, f_2}{u}\bar{u}^2;$$

$$\beta^2 = 4\bar{u}^2\bar{H}_w + (\bar{H}_w - 1)^2;$$

$$\bar{u} = \frac{u}{\sqrt{\{2[H_\infty + (d/c_1)]/A\}}};$$

$$\bar{u}_w = \frac{u}{\sqrt{\{2[H_w + (d/c)]/A\}}};$$

$$\bar{H}_w = \frac{H_w + (d/c)}{H_\infty + (d/c_1)}.$$

From the formulae obtained, in particular cases, the known results are obtained for an incompressible liquid. In the case of a thermo-isolated wing or an axisymmetrical body, where $Pr = Le_i = 1$, the problem is solved simply. Here, it may be assumed that $H = \text{const}$, which allows one to obtain expressions for the friction coefficient and other values, which generalize the solutions known.

For the non-thermo-isolated wing or the axisymmetrical body, the problem of gas motion in a turbulent boundary layer has not been solved.

Lately some works devoted to the consideration of turbulent boundary layer, generalizing the analogy method have been published; it is supposed in these works that the power velocity distribution law is valid in the boundary layer $(v_x/u) = (\eta/\delta)^n$. Of these, Belyanin's work, which was reported at the Congress of Mechanic Workers in January 1960, should be mentioned.

REFERENCES

1. I. P. GINZBURG, *Vestnik Leningradskogo Gosudarstvennogo Universiteta*, N 1 (1961).
2. V. S. AVDUEVSKY and E. I. OBROSKOVA, *Izv. AN S.S.R. Mekhanika i Mashinostroenie*, N 4 (1960).
3. E. C. STUPACHENKO, B. S. DOTSENKO, I. M. STAKHANOV and E. V. SAMUILOV, Fizicheskaya gazodinamika, *Sbornik Izd. AN SSSR, Moskva* (1959).
4. S. CHAPMAN and T. COWLING, *The mathematical theory of non-uniform gases*. Cambridge (1939).
5. R. FAULER and E. GUGENGEIM, *Statisticheskaya termodinamika*. Gosinizdat, Moskva (1949).
6. A. I. GUBANOV and Y. P. LUNKIN, *Zh. Tekh. Fiz.* XXVI, N 11 (1957).
7. M. BODENSTEN, *Roczinike Chemie*, **18**, 374 (1938).
8. G. CARERI, *Nuovo Cimento*, **7**, 155 (1950).
9. A. S. PREDVODITELEV, E. W. STUPACHENKO, J. P. STAKHANOV, E. V. SAMUILOV, A. S. PLESHANOV and J. B. ROGDESTVENSKY, Tablitsa termodinamichekikh funktsii vozdukha v intervale temperatur ot 1000 do 12000° v intervale davlenii ot 0·001 do 1000 atm, *Izd-vo AN SSSR* (1957).
10. MOORE, Reshenie uravnenii laminarnogo pogranichnogo sloya s uchetom dissotsiatsii dlya szhimaemoi zhidkosti s izmenyayuscchimisya fizicheskimi svoistvami, *Mekhanika, Sbornik perevodov*, N 5 (1953).
11. E. P. VAULIN, *Dokl. AN SSSR*, **113**, N 6 (1957).
12. C. COHEN, R. BROMBERG and R. LIPKIS, *VRT*, N 4 (1959).
13. R. E. DORFMAN, N. J. POLSKY and S. W. ROMANENKO, *PMM*, vypusk 2, 22 (1958).
14. S. M. SCALA and C. W. BAULKNIGHT, *VRT*, N 12 (1960).
15. S. LEVY, *JAS*, **21**, N 7 (1954).
16. T. J. LI and H. T. NAGAMATSI, *JAS*, **22**, 607–616 (1955).
17. C. COHEN and E. RESHOTKO, *NACA, Rep.* 1294 (1956).
18. E. RESHOTKO and C. COHEN, *NACA, Tech. Note*, 3513 (1953); 3525 (1955).
19. L. LEES, *Jet Propulsion*, **26**, N 4 (1956).
20. J. A. FAY and F. R. RIDDELL, *JAS*, **25**, N 2 (1958).
21. J. A. FAY, F. R. RIDDELL and N. H. KEMP, *Jet Propulsion*, **27** (1957).
22. KALIKHMAN, *Trudy* NII, N 1 (1947).
23. A. G. HEMMIT, Vliyanie vyazkosti na giperzvukovoe obtekanie ploskoi plastiny s konechnoi perednei kromkoi, *Mekhanika, Sbornik perevodov*, N 5 (1959).
24. N. A. TOLMACHEV, *Vestnik MGU*, v. 172, *Mekhanika*, 5 (1954).
25. B. THWAITES, *Aero. Quart.*, **I**, 245–280 (1949).
26. N. ROTT and L. F. CRABTREC, *J. Aero. Sci.*, **19**, 553–565 (1952).
27. W. HAYES, Voprosy raketnoi tekhniki, *Sbornik perevodov*, N 3 (1957).
28. N. E. KOCHIN and L. G. LOITSYANSKY, *DAN SSSR*, XXXVI, N 9 (1942).
29. V. S. AVDUEVSKY and R. M. KOPYATKEVICH, *Izv. AN SSSR, OTN, Mekhanika i mashinostroenie*, N 1 (1960).
30. H. GÖRTLER, *J. Math. Mech.*, **8**, 1–66 (1957).
31. E. M. SPARROW, *J. Fluid Mech.*, **4**, 321–392 (1958).
32. M. E. SHVETS, O priblizhennom reshenii nekotorykh zadach gidrodinamiki pogranichnogo sloya, *PMM*, 13, vy. p. 3 (1949).
33. Z. S. GALANOVA, *Vestnik LGU*, N 19 (1960).
34. G. A. KULONEN, *Vestnik LGU*, N 19 (1960).
35. R. T. KRAMER and H. M. LIBERSTEIN, *JAS*, N 8 (1959).
36. G. A. KULONEN, *Vestnik LGU*, N 1 (1961).
37. V. V. LUNEV, Primenenie metoda malykh vozmushchenii k zadacham laminarnogo pogranichnogo sloya, *PMM*, vyp. 5, XXI, (1957).
38. C. R. WILKE, *J. Chem. Phys.* **18**, No. 4 (1950).

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Abstract—The paper present equations for a gas mixture flow in laminar and turbulent boundary layers with a discussion of possible methods of their solution.

In particular, some new classes of similar solutions and a scheme for an approximate solution for the equations with the use of the Kochin–Loitsyansky method are mentioned. A series of new results on a turbulent boundary layer is given.

Résumé—L'article donne des équations relatives à un mélange de gaz en écoulement dans les couches limites laminaire et turbulente et discute des méthods possibles pour les résoudre.

En particulier, des classes nouvelles de solutions semblables et le schéma d'une solution approchée utilisant la méthode de Kochin–Loitsyansky y sont mentionnés. Une série de nouveaux résultats sur la couche limite turbulente est donnée.

Zusammenfassung—Für die laminare und turbulente Grenzschichtströmung eines Gasgemisches werden Gleichungen mit möglichen Lösungsmethoden angegeben.

Insbesondere sind neue Arten von Ähnlichkeitslösungen erwähnt und eine Möglichkeit der näherungsweisen Lösung mit Hilfe der Kochin–Loitsyansky Methode. Für die turbulente Grenzschicht ist eine Reihe neuer Ergebnisse angegeben.

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THE HEAT TRANSFER COEFFICIENT FOR FLOW IN A PIPE

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Abstract—The paper records new data on the local heat transfer coefficient in a straight pipe remote from the entrance. Water and air were used, at Reynolds numbers from 300–100 000; the pipes were horizontal, of seven different diameters from 1·27 to 5·08 cm. Temperature differences were small.

Results for laminar and turbulent flow are compared with conventional formulae. For laminar flow they are considerably higher than predicted by elementary theory because of natural convection effects; these are considered in some detail.

A PROGRAMME of experimental work on heat transfer between a pipe and fluid flowing through it has been in progress at NEL for some years. The object has been to study in detail the variation in the local heat transfer coefficient arising from disturbances to the fluid flow caused by an abrupt change in the diameter of the pipe, by a bend or by an abrupt elbow. The fluids used have been air and water. The work has been somewhat similar to that described by Petukhov and Krasnoschekov [1].

As a by-product of these experiments a considerable amount of data has been accumulated on the value of the local heat transfer coefficient at positions remote from such disturbances, where the variation along the pipe has fallen to a negligible level. Data of this type may be compared with the many formulae which are available for calculating the local coefficient under such circumstances, or the average coefficient for a long pipe, and it is felt that, as there does not appear as yet to be any final agreement on the best formulae to be used, the data may be worth putting on record. At the lower Reynolds numbers interesting observations on the effect of natural convection have been made.

The apparatus used in these experiments was in all important characteristics similar to that described in an earlier paper [2]. The various experimental pipes were heated by the passage of electricity along the pipes themselves; pipe temperatures were obtained by using thermocouples attached to the outside at a series of positions along the pipe, the corresponding temperatures at the inside surface being deduced by calculation. A number of thermocouples were attached to the pipe at each position in order to investigate the distribution of temperature around the periphery. For most purposes, the heat transfer coefficients calculated for each position have been based on the average of the readings of the thermocouples at that position. Local bulk fluid temperatures were calculated on the basis of measurements of inlet temperature, rate of flow of fluid, and heat input up to the point in question; outlet temperatures were also measured, and were used as a check on accuracy. Heat fluxes and pipe-to-fluid temperature differences were kept as low as possible, consistent with reasonable accuracy of measurement, in order to avoid complications due to variation of physical properties with temperature. Precautions were taken to avoid errors due to fouling and, when water was used, deposition of bubbles of air. Experiment and calculation were used to make the necessary corrections for the flow of heat from the pipe to the outside atmosphere, and along the heavy copper leads conveying the electric current. Results were obtained with pipes of the following internal diameters: 0·5, 0·6, 0·75, 0·8, 1·0, 1·5 and 2·0 inches (1·27–5·08 cm). The range of Reynolds number explored extended from 100 000 down to 300; for values of Re less than about 5000, however, precision became increasingly difficult to obtain, first because of instability of flow in the neighbourhood of the critical Reynolds number, and, for still lower values of Re, because of effects due to natural convection, including the development of substantial differences in temperature between the top and

bottom of the pipe. With air at low Reynolds numbers, the corrections were of comparable magnitude to the true heat transfer, and consequently no worthwhile results were obtained with laminar flow.

Each observation was reduced to a Nusselt number associated with a Reynolds number and Prandtl number (*Nu*, *Re* and *Pr* respectively), the physical properties having been taken at the bulk temperature. The range of variation of *Pr* in the experiments with air was negligible and a common value of 0·7 may be taken. For the tests with water, values from 4 to 12 were involved, the majority being around 8. Many proposals have been made for empirically representing the effect of the variation of Prandtl number; most involve the use of Pr^n, where n is given various values; for turbulent flow, they are generally in the neighbourhood of 0·4. The procedure has been adopted here of presenting the results in the form of a graph of $Nu/Pr^{0·4}$ against *Re*. Whether or not the choice of $Pr^{0·4}$ is the best possible, it should be sufficient to eliminate the comparatively small effect due to Prandtl number variations in the tests with water. The results are presented in this way in Fig. 1.

We consider first the data obtained with turbulent flow, for Reynolds numbers of about 8000 upwards. It will be seen that the results with water are closely correlated but that those obtained with air are not reconciled with those for water. As already mentioned, there are a variety of formulae which can be compared with

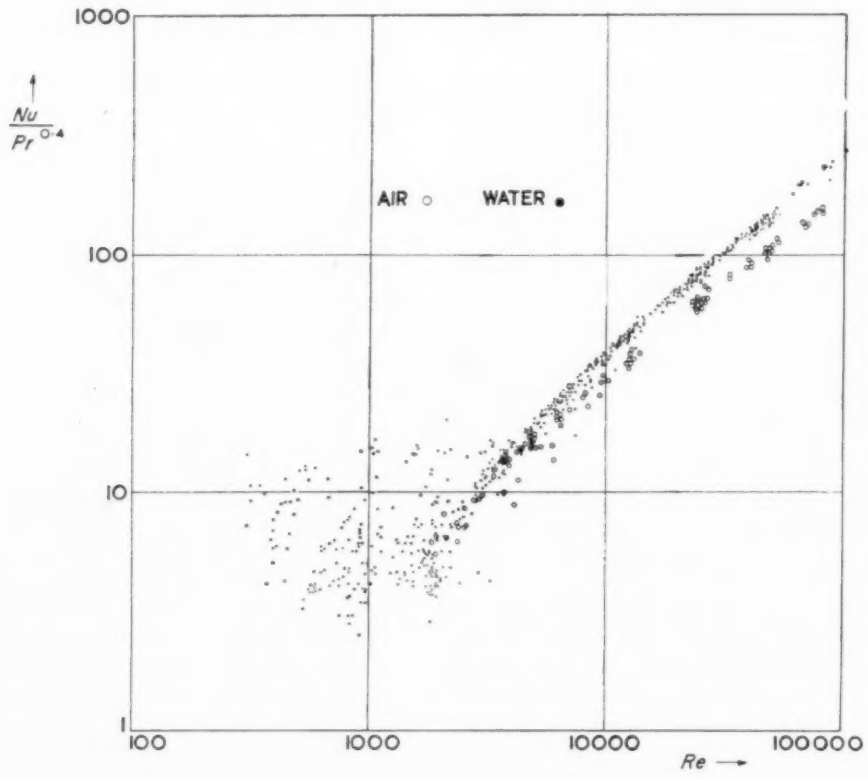

FIG. 1. Correlated experimental results, for air and water, obtained with pipes of diameters varying from 0·5 to 2·0 in. (1·27–5·08 cm).

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these results. This is not the place for an exhaustive analysis, but two important examples will be briefly considered. First, there is the familiar empirical equation [3, p. 145]:

$$Nu = 0{\cdot}023 Re^{0{\cdot}8} \times Pr^{0{\cdot}4},$$

the physical constants being taken at the bulk temperature. A line corresponding to this equation is drawn in Fig. 2, which presents the results for turbulent flow on a larger scale. This equation can evidently serve as a compromise between the two sets of data for water and air, but does not represent either precisely. The slope of the line is also not perfectly in accordance with the results, but within the range of Reynolds numbers investigated the discrepancy is not serious. This type of formula has from its simplicity much to recommend it but, from the present evidence, different values of the coefficient would be required for air and for water, in the neighbourhood of 0·018 and 0·026 respectively. Other workers have proposed 0·021 and 0·027 [4, p. 219]. An improvement could be effected by changing the index of the Prandtl number, but a very substantial change would be needed.

The other formula which will be considered is of a completely different type, more satisfactory in derivation but less convenient in use; it is that due to Martinelli, based on refinements of Reynolds' analogy (see McAdams [4, p. 212]):

$$\frac{Nu}{Re \,.\, Pr} = \frac{\sqrt{(f/2)}}{5q\left\{Pr + \ln(1+5Pr) + 0{\cdot}5Pr \,.\, \ln\left[\frac{Re}{60}\sqrt{\left(\frac{f}{2}\right)}\right]\right\}}$$

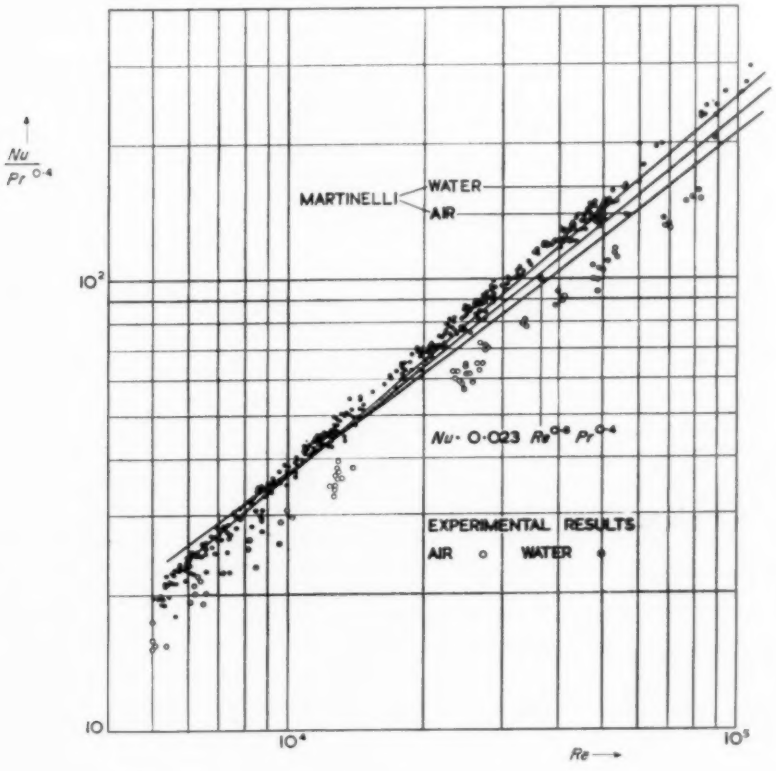

FIG. 2. Results for turbulent flow.

where f is the friction factor, and φ is a temperature-difference ratio. On the present method of plotting, this equation affords a variation with Prandtl number, and in Fig. 2 lines are drawn representing Prandtl numbers of 0·7 and 8·0, i.e. air and an average value for water. The line for water agrees quite well with the experimental data, both in slope and position, over the whole range of Reynolds number. The line for air diverges from that for water in the right direction, and at the higher Reynolds numbers has higher Reynolds numbers: fluctuations were not detected above about $Re = 9000$. At lower Reynolds numbers the extent of the fluctuations increased and reached a magnitude of about ± 10 per cent in the region of $Re = 3000$. Much more stable results were obtained at such Reynolds numbers if the flow was deliberately "tripped" into turbulence by means of an artificial obstruction introduced near the entrance to the pipe. It will be observed from Fig. 1 that the local Nusselt number varies very sharply

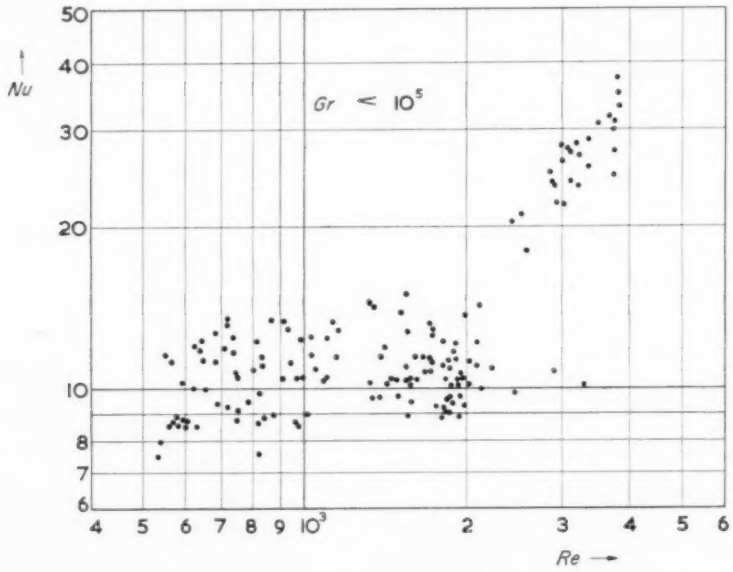

FIG. 3. Results for low Reynolds numbers with the Grashof number less than 10^5.

the right slope. Throughout the range, however, it predicts Nusselt numbers which are a good deal higher than the experimental values, and at the lower end it passes above the line for water, in contradiction to the experimental observations, which still remain below. Similar results are predicted by the various formulae of similar type developed by Kutateladze [3, p. 143].

The results obtained in the transitional region do not lend themselves to detailed analysis because of the instability already referred to. This produced fluctuating temperatures, the extent of the fluctuations being small at the with Reynolds number in this range, in a similar manner to the well-known variation of the average Nusselt number. (See, for example, Fishenden and Saunders [5, p. 110].)

For Reynolds numbers below about 2300 the results were quite stable. As will be seen from Fig. 1, however, the Nusselt numbers obtained varied very widely; two factors which were clearly involved in this variation were the diameter of the pipe and the rate of heat flux, and it is obvious that natural convection was responsible. The scatter is reduced if limits are placed on the magnitude of the Grashof number; Fig. 3 shows the variation of Nu with Re

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for those tests in which the Grashof number was less than 10^5. It appears that there is no consistent variation with Reynolds number until a departure from laminar flow occurs at about $Re = 2300$; this in accordance with theory which indicates (in the absence of natural convection) a constant value of 4·36 for the Nusselt number. The values measured are, however, in the region of 10, which is much greater than the theoretical figure, and indicates that even at the lowest range of Grashof number encountered the effect of natural convection is very pronounced.

The predominance of Grashof number effects and the probable lack of variation with Reynolds number suggest that better correlation of the laminar flow results would be provided by plotting *Nu* against *Gr*. This has been done in Fig. 4, which shows that this method of correlation is reasonably successful. The results may

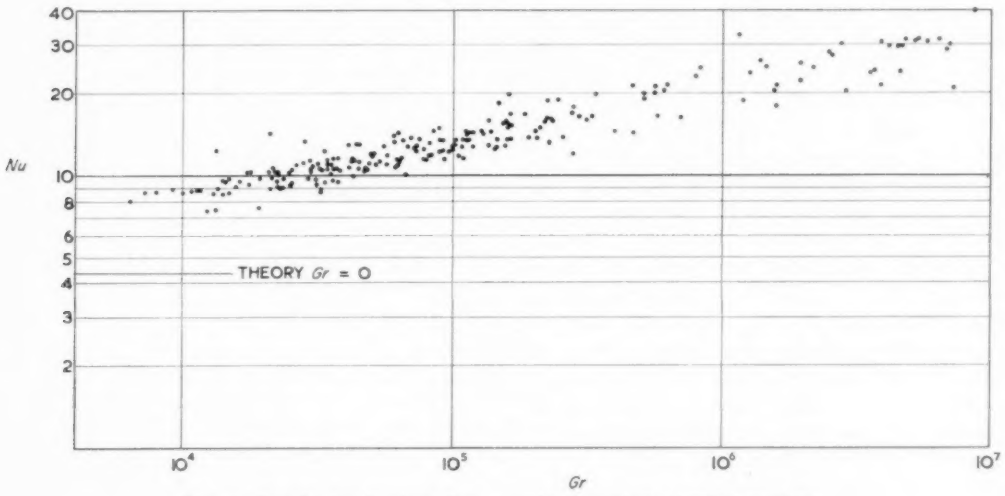

FIG. 4. Results for laminar flow correlated against Grashof number.

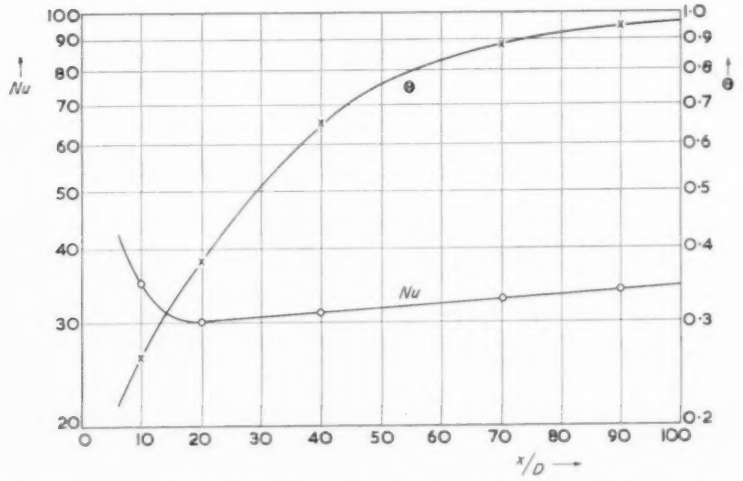

FIG. 5. Variation of Nusselt number and top/bottom temperature difference along a pipe, at a Reynolds number of 2100.

be approximately represented by the equation $Nu = 4{\cdot}36\ (1 + 0{\cdot}06\ Gr^{0{\cdot}3})$.

Another feature of some interest which arises at low Reynolds numbers is the development of a substantial difference in temperature between the top and bottom of the experimental pipe, indicating a gradual stratification of the warmer water above the cooler water in the pipe. In Fig. 5 the Nusselt number and the top-to-bottom temperature difference, θ, (in dimensionless form), are plotted against position along the pipe (measured in diameters), showing how the stratification effect develops and revealing that it has little influence on the value of the Nusselt number. The magnitude of the effect is governed by the dimensions of the pipe and the thermal conductivity of the material of which it is made, since the flow of heat by conduction in the pipe wall tends to reduce it. If the same temperature difference, θ, is plotted against the Grashof number, the results fall into compact groups according to the particular pipes with which they were obtained. For a given limited range of Grashof number a rough correlation can be made between θ and a figure indicating the amount of heat which would flow from top to bottom of unit length of the pipe under unit temperature difference.

The writer is not aware of any theoretical or experimental work which can be directly compared with the present data for laminar flow. Qualitatively, of course, the results have been familiar for many years. Morton [6], deals specifically with this problem, but his work is concerned with very small Grashof numbers.

ACKNOWLEDGEMENT

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REFERENCES

1. B. S. Petukhov and E. A. Krasnoschekov, Heat exchange in the initial part of a tube when there is a mixed boundary layer. In: *Teploperedacha i Teplovoe Modelirovanie*, Editor, M. A. Mikheev, pp. 187–200, Izdatel'stvo Akademii Nauk SSSR, Moscow (1959).
2. A. J. Ede, C. I. Hislop and R. Morris, Effect on the local heat-transfer coefficient in a pipe of an abrupt disturbance of the fluid flow: abrupt convergence and divergence of diameter ratio 2:1. *Proc. Instn Mech. Engrs, Lond.*, **170**, (38), 1113–1130 (1956).
3. S. S. Kutateladze, *Osnovy Teorii Teploobmena*. Mashgiz, Moscow (1957).
4. W. H. McAdams, *Heat Transmission* (Third Ed.). McGraw-Hill, New York, London (1954).
5. M. Fishenden and O. A. Saunders, *An Introduction to Heat Transfer*. Clarendon Press, Oxford (1950).
6. B. R. Morton, Laminar convection in uniformly heated horizontal pipes at low Rayleigh numbers. *Quart. J. Mech. Appl. Math.*, **12**, (4), 410–420 (1959).

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Résumé—L'article rassemble des données nouvelles sur le coefficient de transmission de chaleur dans un tuyau droit loin de l'entrée. L'étude a été faite avec de l'air et de l'eau, des nombres de Reynolds de 300 à 100.000, des tuyaux horizontaux de diamètres échelonnés entre 1,27 et 5,08 cm. Les différences de températures étaient petites.

Les résultats des écoulements laminaire et turbulent sont comparés à ceux que donnent les formules habituelles. Pour un écoulement laminaire, ils sont considérablement plus élevés que ceux prévus par la théorie à cause des effets de convection libre, ceci est étudié en détail.

Zusammenfassung—Für den örtlichen Wärmeübergangskoeffizienten in einem geraden Rohr, in einiger Entfernung vom Einlauf werden neue Daten gebracht. Als Versuchsmedium diente Wasser und Luft bei Reynoldszahlen von 300–100 000 in waagrechten Rohren mit sieben verschiedenen Durchmessern von 1,27 bis 5,08 cm. Die Temperaturdifferenzen waren klein.

Die Ergebnisse für laminare und turbulente Strömung wurden mit bekannten Formeln verglichen. Für laminare Strömung liegen diese Ergebnisse beträchtlich über den Werten der Elementartheorie. Dies rührt vom Einfluss der natürlichen Konvektion her und wurde ebenfalls untersucht.

Аннотация—В статье излагаются результаты экспериментальных исследований по определению коэффициента теплообмена при течении в трубах воды и воздуха в диапазоне изменения числа Рейнольдса от 300 до 100 000.

Проведенные исследования показали, что при $Re > 8000$ опытные данные для воды и воздуха не согласуются между собой. При $Re < 8000$ опытные данные для воды и воздуха могут быть описаны одной приближенной формулой.

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TRANSFER PROCESSES IN A FREE (JET) TURBULENT BOUNDARY LAYER

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Аннотация—Излагается метод решения тепловых и динамических задач струйных течений путём сведения дифференциальных уравнений пограничного слоя к эквивалентному уравнению теплопроводности. Приводится физическое обоснование метода, сводящееся к переносу процесса в фиктивное линейное пространство, связь которого с реальным полем течения определяется из опыта по зависимости скорости на оси струи от координаты.

Излагаются результаты специально поставленных опытов по изучению механизма смешения неизотермических струй сжимаемого газа. Результаты показывают определяющую роль градиента плотности потока импульса в процессах переноса в турбулентных струях.

(1) PHENOMENA of turbulent transfer in free flow of an incompressible liquid or compressible gas form the working basis for different technical devices. They are extremely important in furnaces, stove techniques, industrial ventilation, etc. For this reason the experimental and theoretical study of heat, matter and impulse transfer processes in various jet flows is of considerable interest.

Because of insufficient development of the general theory of turbulent flow, different semi-empirical methods have become widely used in studying turbulent jets. One of these is the calculation of free turbulent flow by substituting boundary layer differential equations for equations of heat conductivity. This method, suggested several times in [1, 2, etc.], was widely developed in investigations carried out at the Institute of Energetics of the Academy of Science of the Kazakh S.S.R. and at the Kazakh State University [3, 5]. These investigations were concentrated on such flows as submerged jets of a finite dimension, ets in concurrent and counter flows, etc. Considerable attention was paid to the study of the mixing mechanism in turbulent flow.

(2) The method being developed for solving jet problems is connected with both the external resemblance of integral curves of velocity distribution in a jet and with temperature of a heat source in a solid.

The successful application of this method depends on the fulfillment of certain conditions. Firstly, the flow must depend on the right class of boundary layer problems (i.e. when the transverse velocity component is small compared with the longitudinal component, and the longitudinal transfer is small compared with the transverse transfer). Secondly, motion must take place well away from solid walls and be practically isobaric (i.e. flow lines relatively straight). Thirdly, the initial impulse distribution (heat content), may be arbitrary and this is one of the significant advantages of this calculation scheme.

Under these conditions the phenomenon of jet propagation, from the physical point of view, is reduced to level initial disturbances (impulse, heat flow, etc.).

The jet propagation process is described by non-linear boundary layer differential equations. Although at a distance from the mouth the flow attains steady state, and motion and energy equations are reduced to ordinary differential equations, the transition to steady state motion will be non-linear and dependent on initial conditions. Let us assume that it is possible to bring about the deformation of a real space (x, y) into a fictious space (τ, q) where the process of jet dispersion may be described by linear differential equations.

As further investigations showed, in jet problems it is sufficient to deform only the longitudinal co-ordinate x.

In this case heat and impulse transfer equations in the new deformed space will take the form:

$$\frac{\partial u}{\partial \tau_i} = \frac{1}{y^k}\frac{\partial}{\partial y}\left(y^k \frac{\partial u}{\partial y}\right). \tag{1}$$

Here $k = 0$ and corresponds to a flat jet; $k = 1$, to the axisymmetric jet; $i = q$; $u = \rho u^2$ corresponds to the dynamic and $i = \tau$; $u = \rho u c_p \Delta T$, to the heat problem.

The law of transformation of values of $\tau_d(x)$ and $\tau_h(x)$ is not known in advance but may be obtained from comparison with experimental results, e.g. by comparing calculation and experimental data for the change along the axis of a jet.

In a dynamic problem the co-ordinate $\tau_h(x)$ differs from the corresponding dependant $\tau_d(x)$ because of the difference in thickness of dynamic and thermal mixing regions.

In the steady state region of motion proceeding from dimensionality for $\tau_d(x)$ and $\tau_h(x)$ we obtain:

$$\tau_d = c_d x^2, \tag{2}$$

$$\tau_h = c_h x^2. \tag{3}$$

In this case the ratio

$$\frac{\tau_d}{\tau_h} = \frac{c_d}{c_h} = \sigma_t \tag{4}$$

indicates the turbulent Prandtl number. As further investigation showed (Fig. 1), the value σ_t determined by equation (4) coincides with the numerical value for the ordinary Prandtl number.

Thus, in the steady state region as well as in the theoretical "mixing lengths" it is necessary to determine only one experimental constant. The developed equivalent heat conductivity problem method makes it possible to calculate the continuous deformation of velocity and temperature profiles with arbitrary initial distribution.

Thus, we succeeded in solving a number of problems (e.g. jet of a finite dimension, a system of jets, etc.), whose solution by other methods is cumbersome or completely impossible.

(3) Now let us consider problems of propagation of submerged jets with a complicated initial velocity (temperature) distribution.

Referring to [3–5], we shall dwell only on the following:

Table 1 gives the basic types of jets investigated, initial conditions and formulae for calculating velocity and temperature fields

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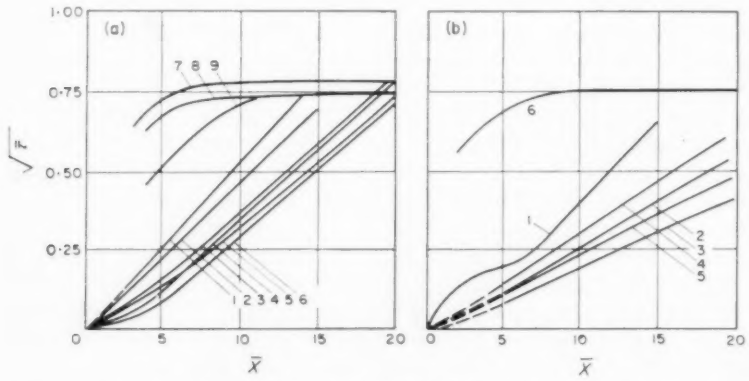

FIG. 1. Dependence $\tau(x)$ for jets investigated:
(a) 1—$\sqrt{\bar{\tau}}$; 2—$\sqrt{\bar{\tau}_d}$—flat jet (ρu_0^2 = const); 3—$\sqrt{\bar{\tau}_h}$; 4—$\sqrt{\bar{\tau}_d}$ axisymmetrical jet (ρu_0^2 = const); 5—$\sqrt{\bar{\tau}_h}$; 6—$\sqrt{\bar{\tau}_d}$ axisymmetrical jet $\{\rho u_0^2 \sim [1 - (y/r_0)]^{2/7}\}$; 7—$\bar{\tau}_d/\tau_h$ flat jet; 8—$\bar{\tau}_d/\tau_h$—axisymmetrical jet (ρu^2 = const); 9—$\tau_d/\bar{\tau}_h$—axisymmetrical jet $\{\rho u_0^2 = [1 - (y^0/r_0)]^{2/7}\}$.
(b) 1—two concurrent flat jets $[(u_2/u_1) = 2]$; 2,3,4,5—jet in a concurrent flow where 2—$\sqrt{\bar{\tau}_d}$ $[(u_f/u_j) = 0{\cdot}143]$; 3—$\sqrt{\bar{\tau}_d}$ $((u_f/u_j) = 0{\cdot}295)$; 4—$\sqrt{\bar{\tau}_h}$ $[(u_f/u_j) = 0{\cdot}143]$; 5—$\sqrt{\bar{\tau}_d}$ $[(u_f/u_j) = 0{\cdot}295]$; 6—$\bar{\tau}_d/\tau_h$ for jet in concurrent flow.
$\bar{\tau} = \tau/e$; $\bar{x} = x/e$ (e—is $\frac{1}{2}$ width of nozzle for flat jet, diameter of nozzle for axisymmetrical jet).

Table 1

Kind of jet	Initial conditions	Solution	Flow scheme
Flat jet	$\lvert y\rvert \leqslant b$; $(\rho u^2)_0 = \text{const}$; $(\rho u c_p \Delta T)_0 = \text{const}$; $\lvert y\rvert > b$; $(\rho u^2)_0 = 0$	$\bar{u} = \frac{1}{2}\left[\psi\left(\frac{y+b}{2\sqrt{\tau}}\right) - \psi\left(\frac{y-b}{2\sqrt{\tau}}\right)\right]$;	
Two flat concurrent jets of different velocity	$-\infty < y < b_2$; $(\rho u^2)_0 = 0$; $-b_2 < y < 0$; $\rho u^2 = (\rho u^2)_2$; $0 < y < b_1$; $\rho u^2 = (\rho u^2)_1$; $b_1 < y < \infty$; $\rho u^2 = 0$.	$\bar{u} = \frac{1}{2}\left[\psi\left(\frac{y}{2\sqrt{\tau}}\right) - \psi\left(\frac{y-b_1}{2\sqrt{\tau}}\right) + \frac{1}{2}\frac{(\rho u^2)_2}{(\rho u^2)_1}\left[\psi\left(\frac{y+b_2}{2\sqrt{\tau}}\right) - \psi\left(\frac{y}{2\sqrt{\tau}}\right)\right]\right]$	
Two flat concurrent jets at different distances from each other	$-\infty < y < z$; $\rho u^2 = 0$; $-z < y < d$; $\rho u^2 = \text{const}$; $-d < y < d$; $\rho u^2 = 0$; $d < y < z$; $\rho u^2 = \text{const}$; $z < y < \infty$; $\rho u^2 = 0$.	$\bar{u} = \frac{1}{2}\left[\psi\left(\frac{y+Z}{2\sqrt{\tau}}\right) - \psi\left(\frac{y+b}{2\sqrt{\tau}}\right) + \psi\left(\frac{y-b}{2\sqrt{\tau}}\right) - \psi\left(\frac{y-Z}{2\sqrt{\tau}}\right)\right]$	
Axisymmetrical jet	$0 \leqslant y \leqslant r_0$; $(\rho u^2)_0 = \text{const}$ $(\rho u c_p \Delta T)_0 = \text{const}$; $\lvert y\rvert > r_0$; $\rho u^2 = 0$.	$\bar{u} = \frac{1}{2\pi\tau}\int_0^{\tau_0}\rho\,\mathrm{d}\rho\int_{-\pi/2}^{\pi/2}\exp[-(\rho^2+y^2-2y\rho\sin\varphi)/4\tau]\,\mathrm{d}\varphi$	
Axisymmetrical jet	$\lvert y\rvert < r_0$; $(\rho u^2)_0 \sim [1-(y/r_0)]^{2/7}$ $(\rho u c_p \Delta T)_0 = \text{const}$; $\lvert y\rvert > r_0$; $\rho u^2 = 0$.	$\bar{u} = \frac{1}{2\pi\tau}\int_0^{\tau_0}\left(1-\frac{\rho}{r_0}\right)^{2/7}\rho\,\mathrm{d}\rho \int_{-\pi/2}^{\pi/2}\exp[-(\rho^2+y^2-2y\sin\varphi)/4\tau]\,\mathrm{d}\varphi$	
Jet flowing from two concentric nozzles	$r_2 < y < r_1$; $(\rho u^2)_0 = (\rho u^2)_2$; $r_1 < y < 0$; $(\rho u_2)_0 = [1-(y/r_0)]^{2/7}$; $\lvert y\rvert > r_2$; $\rho u^2 = 0$.	$\bar{u} = \frac{1}{2\pi\tau}\int_0^{\tau_1}\left(1-\frac{\rho}{r_0}\right)^{2/7}\rho\,\mathrm{d}\rho \int_{-\pi/2}^{\pi/2}\exp(\rho^2+y^2-2y\rho\sin\varphi/4\tau)\,\mathrm{d}\varphi + \frac{1}{2\pi\tau}\frac{(\rho u^2)_2}{(\rho u^2)_1}\int_{-\pi/2}^{\pi/2}\exp(\rho^2+y^2-2y\rho\sin\varphi/4\tau)\,\mathrm{d}\varphi$	

determined with the solution of equation (1). The symbols used here are as follows:

$$\bar{u} = \frac{\rho u^2}{(\rho u^2)_0}$$

or

$$\bar{u} = \frac{\rho u c_p \Delta T}{(\rho u c_p \Delta T)_0}$$

represent the dynamic or heat problem respectively; $(\rho u^2)_0$; $(\rho u c_p \Delta T)_0$ are initial values of impulse density flow and heat content; b_i is the nozzle half width for a flat jet flow; r_i is the nozzle radius for an axisymmetrical jet flow; L is the distance between nozzles for two flat jet flow.

$$q(z) = \frac{2}{\sqrt{(\pi)}}\int_o^z \exp(-t^2)\,\mathrm{d}t$$

is the error integral; φ is the variable angle of integration for sources in a cylindrical system of co-ordinates.

In the cases considered the experimental flow investigation was carried out on special stands, attention being centred mainly on obtaining a detailed picture of flow on the basis of analysis of velocity and temperature fields. For the theoretical solution hydrodynamic and hydrostatic integrators were widely used, application of this method to the solution of jet problems is described in [6, 7].

Regime and geometrical characteristics of the flows investigated are given in Table 2.

Table 2

Kind of flow	r_1 (b_1) (mm)	r_2(b_2) (mm)	u_1 (m/s)	u_2 (m/s)	u_1/u_2	z_1 (mm)	ΔT_0 (T_0–T_{average})
Flat jet	8·4	—	39·0	—	—	—	10–15°
Two flat concurrent jets of different velocity	10	10	77·6; 74·3; 35·0; 15·2	38·8; 26·5; 17·5; 30·4	0·5; 2·0; 2·8	0	—
Two flat concurrent jets at different distances from each other	7·0	7·0	80·8	80·0	1	34·8 25·5 17·5 3·5	
Axisymmetrical jet with uniform initial velocity distribution	5·0	—	60·0	—	—	—	20–25°
Axisymmetrical jet with initial velocity distribution of type	6·3	—	40·0	—	—	—	20–25°
Axisymmetrical jet flowing from two concentric nozzles	20·75	39·25	—	—	1·0		
	20·75	39·25			2·0		
	20·75	39·25			3·0		
	20·75	53·0			1·0		
	20·75	53·0			2·0		
	20·75	53·0			3·0		
	20·75	67·0			1·0		
	20·75	67·0			2·0		
	20·75	67·0			3·0		
	2·0	6·75			1·0		
	2·0	6·75			2		
	2·0	6·75			3		

Figure 1 shows dependants $\tau_d(x)$ and $\tau_h(x)$ for the flow cases investigated. Velocity and temperature profiles in transverse sections of the jets were calculated by these dependents. The calculation and experimental data are in good agreement.

(4) The application of the developed calculation method of jet problems to complicated turbulent flows, in particular, to a jet propagating in a concurrent flow is of considerable interest.

A detailed discussion of this problem is given in [8]. It is worthwhile to note that because of linearity and uniformity of equation (1) it may be easily generalized for motion of a surrounding medium with a constant velocity. In this case value u in equation (1) should mean $\rho u^2 - (\rho u^2)_f$ or $\rho u c_p \Delta t$ for dynamic and heat problems respectively. Here $(\rho u^2)_f$ is the value of impulse density flow in a surrounding flow.

Although formally the complicated flow problem is reduced to corresponding problems for a submerged jet, differences in boundary conditions leading to boundary distortions, etc. influence the type of dependent values τ_h and τ_d upon x (Fig. 1b). Otherwise the solution of the problem is exactly the same as the solution for the submerged jet under equal initial conditions (see Table 1).

The distribution of a slightly heated jet in a concurrent uniform flow was investigated experimentally. Regime and geometrical characteristics of the flow investigated are given in Table 3.

Dependents $\tau_d(x)$ and $\tau_h(x)$ obtained in an analogous way to that described above are shown in Fig. 1(b). Comparison of the experimental and calculation data obtained for impulse and temperature flow density distribution

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Table 3

d_0 jet (mm)	d_0 jet (m/s)	u_f (m/s)	$\frac{u_f}{u_j} = m$	ΔT_0 jet (°C)
20	60	0	0	20–25
20	42	6	0·143	20–25
20	42	12·4	0·295	20–25
20	42	15·8	0·380	20–25

in transverse jet sections (Fig. 2) showed their satisfactory convergence.

(5) Another type of complicated turbulent flow studied is the jet propagation in a counter uniform flow. Such flows are often met in practice, and apparently may find a wide application as aerodynamic flame stabilizers [9, 10].

While referring for details to [9, 11, 12, 13] we noted the following.

The type of flow considered has some peculiarities which make it different to ordinary problems of boundary layer theory. In particular, in such a flow there is a considerable static pressure gradient (up to 30 per cent from the initial jet head velocity) both in its longitudinal and transverse directions. In addition, the transverse velocity component which at the end of the jet greatly exceeds the longitudinal component becomes a considerable value.

Hence, the calculation carried out by the method of substitution of the equivalent heat conductivity problem in this case gives only a correct qualitative flow picture (presence of zero surface velocities, closed circulation zones, etc.) but does not give any satisfactory quantitative agreement with experiment. In the work carried out, fields of static pressure, velocity and excessive temperature (above the surrounding flow) at different values of the basic parameter $m = (u_f/u_{0j})$ were experimentally studied.

Regime and geometrical characteristics found in experiments when studying a flat counter jet are listed in Table 4.

(6) The investigation of heat and mass transfer, etc. in a jet is of considerable interest for studying flows. In a number of works (see, e.g. Ref. 6) it was assumed that impulse flow density gradient ρu^2 plays a determinative role in transfer processes. To verify this assumption special experiments were carried out in which the attenuation of a compressed gas

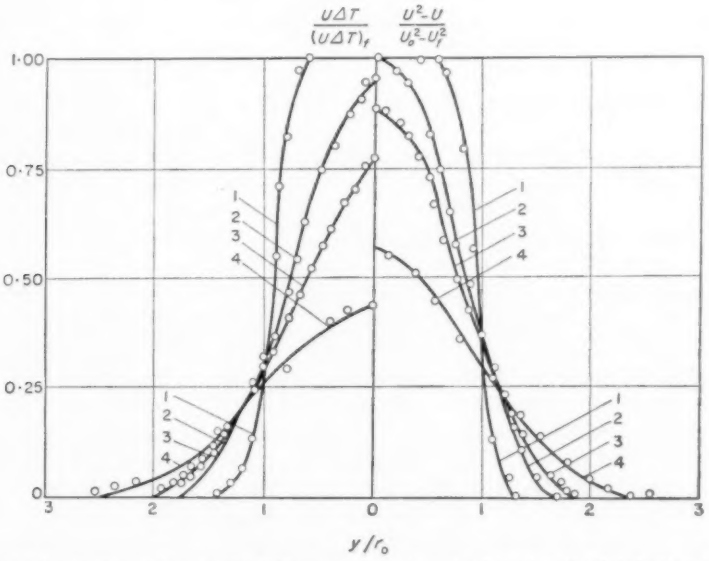

FIG. 2. Comparison of experimental and calculation results on impulse flow density distribution and heat content in transverse sections of jet propagating in concurrent flow: 1—$x/d_0 = 2$; 2—$x/d_0 = 7$; 3—$x/d_0 = 10$; 4—$x/d_0 = 15$; ○—experiment; ———— calculation on a hydrogenerator.

Table 4

Nozzle width (mm)	Regime m	1	2	3	4	5	6	7	8
		0·2	0·25	0·3	0·35	0·40	0·45	0·5	0·6
6	u_f	8·0	10·0	9·0	14·0	16·0	18·0	16·0; 25·0	20·0
	u_{0j}	40·0	40·0	30·0	40·0	40·0	40·0	32·0; 50·0	33·3
12	u_f	8·0		9·0				10·0; 16·0; 25·0	20·0
	u_{0j}							20·0; 32·0; 50·0	33·3
16	u_f			9·0		16·0	18·0	10·0; 16·0; 25·0; 30·0	20·0
	u_{0j}			30·0		40·0	40·0	20·0; 32·0; 50·0; 60·0	33·3
	ΔT_0			22·0		22·0		12·0; 22·5; 29·5; 12·0	
								22·0; 36·0	18·0
								36·0	

Table 5

Jet			Concurrent flow			Relation		
T_0(°K)	u_0 (m/s)	$\frac{\rho_0 u_0^2}{2}$ (mm water column)	T_0 °K flow	u_f (m/s)	$\frac{\rho_f u_f^2}{2}$ (mm water column)	$\frac{T_0}{T_f} = w$	$\frac{u_f}{u_0} = m_f$	$\frac{(\rho u^2)}{(\rho u^2)_0} = m$
320	17–60	16–200	300	0–29·5	0–48	1·08	0–1·67	0–3
600	24–87	16–200	300	0–29·5	0–48	2·0	0–1·22	0–3
900	30–105	16–200	300	0–29·5	0–48	3·0	0–1·0	0–3

propagating in a concurrent uniform flow was investigated.

Limits of the change of parameters of a jet and flow as well as their relations are given in Table 5.

As a basic value characterizing transfer intensity in a jet, the relation of temperature exceeding the surrounding flow at a fixed point on the axis of the jet to initial temperature of the jet ($\Delta T_m/\Delta T_0$) was chosen.

The change of this relation at various ratios of jet and flow dynamic heads gives a vivid representation of transfer intensity in a jet.

Figure 3 shows the change of this relation depending on the parameter m (see Table 5) for the greatest parameter value $w = 3$ obtained in our experiments. As is seen from this figure, the maximum value of ($\Delta T_m/\Delta T_0$), and, consequently, the minimal mixing practically coincide with the value $m = 1$ (at equal values of dynamic heads in a jet). Relation of velocity values (u_f/u_0) and mass velocity $(\rho u)_f/(\rho u)_0$, are also given for comparison along the abscissa axis.

As is seen from this figure, at equal velocity values in the jet and the flow or at those of the mass velocity the mixing intensity is noticeably higher than at equal values of the dynamic heads.

The experimental results obtained confirm the assumption about the determinative role of the impulse flow density gradient in transfer processes and also substantiated the choice of impulse flow density, heat content, etc.

(7) In obtaining transfer processes in turbulent jets, the so-called turbulent Prandtl number σ_t determined as the relation of turbulent impulse and heat transfer coefficients [$\sigma_t = (\nu_t/a_h)$] plays a significant role.

Knowing the above mentioned number allows the calculation of a heat problem by known

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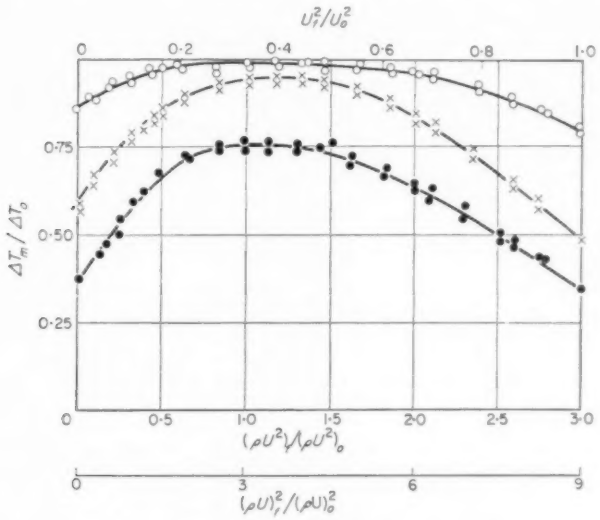

FIG. 3. Dependence $\Delta T_m/\Delta T_0$ on $(\rho u^2)_r/(\rho u^2)_0$ [as well as on u_r^2/u_0^2 and $(\rho u)_r^2/(\rho u_0^2)$] 0 — $x/d_0 = 2{\cdot}5$; $x - x/d_0 = 5$; 0 — $x/d_0 = 7{\cdot}5$.

results obtained theoretically or experimentally for a dynamic problem.

As is known from the literature, the number σ_t has the value $\sim 0{\cdot}72$–$0{\cdot}75$ in free turbulent jets. However, the determination of this number was carried out in a great majority of the cases with an air motion where this number is near to the physical meaning of the Prandtl number [$\sigma = (\nu/a) = 0{\cdot}72$].

In this direction special experiments for determining σ_t in submerged axisymmetrical slightly heated turbulent jets of high viscosity at the physical Prandtl number of $\sim 10^3$ (transformer oil) were carried out at the Institute of Energetics of the Academy of Science of the Kazakh S.S.R. Fields of the dynamic head and temperature in various jet sections were studied in the experiments.

To determine σ_t the relation between temperature distribution and velocity distribution was used from the known free turbulence theory

$$\frac{\Delta T}{\Delta T_m} = \left(\frac{u}{u_m}\right)^{\sigma_t}. \tag{5}$$

The value σ_t determined from experimental data with the help of formula (5) proved to be 0·72 and constant along the jet section.

Thus, value σ_t is confirmed as a purely hydrodynamic characteristic of the flow and does not depend upon the physical properties of the liquid.

REFERENCES

1. N. REICHARDT, Gesetzmassigkeiten der freien Turbulenz, *Forschungsh Ver. Dtsch. Ing.*, 414, (1942).
2. P. V. MELENT'EV, *Nauchnye trudy Stalingradskogo mekhanicheskogo instituta*, tom 1, (1952).
3. L. A. VULIS, *Izv. An Kazakh. S.S.R.*, seria energeticheskaya, N 2, **18** (1960).
4. B. P. USTIMENKO, *Izv. AN Kazakh. S.S.R.*, seria energeticheskaya, N 2, **18** (1960).
5. I. B. PALATNIK, *Izv. AN Kazakh. S.S.R.*, seria energeticheskaya, N 2, **18** (1960).
6. *Sbornik*, Issledovanie fizicheskikh osnov rabochego protsessa topok i pechei (Edited by L. A. VULIS), *Izdatel'stvo AN Kazakh. S.S.R.* (1957).
7. V. G. BESPALOVA, *Izv. AN Kazakh. S.S.R.*, seria energeticheskaya, N 2, **16** (1960).
8. I. B. PALATNIK, *Izv. AN Kazakh. S.S.R.*, seria energeticheskaya, N 2, **18** (1960).
9. L. A. VULIS and T. P. LEONT'EVA, *Izv. AN Kazakh. S.S.R.*, seria energeticheskaya, N 9 (1955).
10. L. A. VULIS, T. P. LEONT'EVA and A. V. TONKONOGY, *Vestnik AN Kazakh. S.S.R.*, N 5 (1954).
11. L. A. VULIS, V. P. KASHKAROV and T. P. LEONT'EVA, Issledovanie fizicheskikh osnov rabochego protsessa topok i pechei, *Izdatel'stvo AN Kazakh. S.S.R.* (1957).
12. T. P. LEONT'EVA and B. P. USTIMENKO, *Trudy Instituta energetiki AN Kazakh. S.S.R.*, tom III.
13. A. SCHAFFER and A. B. CAMBEL, *Jet Propulsion*, **25**, (1955); **26** (1956).

Abstract—The method of solution of heat and dynamic problems for jet flows is given by means of reduction of boundary layer differential equations to an equivalent heat conduction equation. A physical substantiation of the method is presented, reduced to the transfer of the process into a fictitious linear space, the connexion of which with the real field of a flow is determined experimentally according to the dependence of the velocity at the axis of the jet on the co-ordinate.

Results of special experiments on studying the mixing mechanism of non-isothermal jets of a compressible gas are adduced. They show the determinative role of the impulse flow density gradient for transfer processes in turbulent jets.

Résumé—Cet article présente une méthode de résolution des problèmes dynamique et thermique dans les jets, elle consiste à réduire les équations différentielles de la couche limite en une équation de conduction thermique équivalente.

L'auteur donne une vérification physique de la méthode en transposant son problème dans un espace linéaire fictif. La relation entre cet espace et le champ réel de l'écoulement est déterminée expérimentalement d'après la dépendance de la vitesse axiale du jet sur l'autre coordonnée.

Les résultats d'expériences particulières sur le mécanisme du mélange de jets non-isothermes de gaz compressible sont présentés. Ils montrent le rôle déterminant du gradient de densité sur les phénomènes de transport dans les jets turbulents.

Zusammenfassung—Als Lösungsmethode wärmetechnischer und dynamischer Probleme des turbulenten Freistroms wird eine Verkürzung der Differentialgleichungen der Grenzschicht in äquivalente Gleichungen für Wärmeleitung angegeben. Um die physikalische Gültigkeit der Methode zu beweisen, wird der Prozess in einen fiktiven linearen Raum übertragen und die Verbindung mit dem wirklichen Strömungefeld experimentell hergestellt; dabei soll die Geschwindigkeit in der Achse von der Koordinate abhängen.

Die Ergebnisse spezieller Versuche über den Mischmechanismus nicht-isothermer Strahlen eines kompressiblen Gases sind angegeben. Sie zeigen den massgebenden Einfluss des Gradienten der Impulsstromdichte auf Übertragungsvorgänge im turbulenten Freistrom.

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SUBLIMATION OF A SOLID NEAR A CRITICAL POINT IN FLAT AND AXISYMMETRICAL GAS FLOWS

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Аннотация—В работе получено точное решение задачи о сублимации твердого тела в окрестности критической точки в плоском и осесимметричном потоках несжимаемого газа. Установлены конечные соотношения для определения концентрации и температуры на волне сублимации.

Аналогичную задачу рассматривает Робертс [1]. Однако, им строится приближённое решение с использованием интегрального метода Польгаузена (распределение функций в пограничном слое задаётся).

NOMENCLATURE

x, y,	co-ordinates;
u, v,	velocity components;
t,	time;
ρ, P,	density and pressure;
μ, λ, $\mathscr{D}_{1,2}$,	coefficients of viscosity, thermal conductivity and diffusion;
c_p,	specific heat capacity;
$\tilde{\mathscr{D}}$,	velocity of sublimation front propagation;
l,	latent sublimation heat;
$\sigma = \frac{c_p\mu}{\lambda}$,	Prandtl number;
$Sc = \frac{\mu}{\rho\mathscr{D}_{1,2}}$,	Schmidt number;
$Le = \frac{\lambda}{\rho c_p \mathscr{D}_{1,2}}$,	Lewis number;
$H = h + \frac{u^2}{2}$,	complete enthalpy;
$h = Ch_a + (1 - C)/h_b$,	mixture heat content;
C,	mass concentration of subliming vapours in gas mixture;
β,	proportionality coefficient in law of external flow velocity;
R,	gas constant.

Subscripts

1,	in a solid;
a, b,	pertaining respectively to vapour and gas;
0,	on sublimation wave;
B,	at moment of boiling.

1. STATEMENT OF PROBLEM. SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

THE interaction of a gas flow and solid near a critical point, resulting in the sublimation of the solid surface, is considered. The sublimation front is assumed to propagate into the body with constant velocity $\tilde{\mathscr{D}}$: i.e. a stationary sublimation regime is considered.

Equations for a gas and solid are considered to be non-stationary. The solution is given for a solid in a flat and axisymmetrical gas flow.

A heat transfer process takes place both in the gas and the solid. The equation of a gas and solid should therefore be solved simultaneously combining corresponding solutions on a sublimation wave.

The consideration of a diffusion equation is caused by the fact that a bi-component mixture (vapour–gas) is formed during the sublimation of a solid surface.

The problem is solved by assuming a laminar boundary layer. To simplify this problem the

properties of the gas and solid are assumed to be constant.

The following system of co-ordinates is chosen: axis x is directed along the surface of the solid, the critical point being taken as the origin; axis y is normal to the surface of the solid, and is directed towards the gas-side, the value being read from the initial boundary between the gas and solid (Fig. 1).

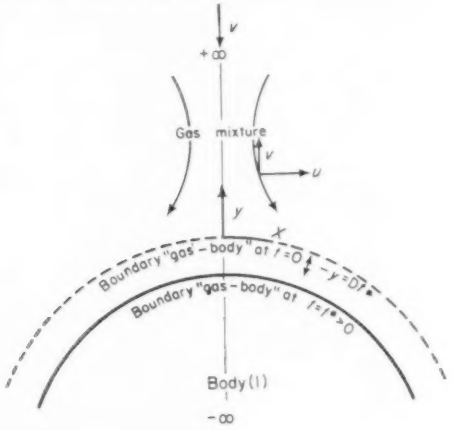

FIG. 1.

The problem considered is described by an original system of equations consisting of equations of continuity, motion, diffusion and energy for a gas mixture.

$$\left.\begin{aligned}
&\frac{1}{x^{n-1}}\,\frac{\partial(ux^{n-1})}{\partial x}+\frac{\partial v}{\partial y}=0;\\
&\rho\,\frac{\mathrm{d}u}{\mathrm{d}t}=-\frac{\partial P}{\partial x}+\mu\,\frac{\partial^2 u}{\partial y^2}\\
&\frac{\partial P}{\partial y}=0;\\
&\rho\,\frac{\mathrm{d}C}{\mathrm{d}t}=\rho\mathscr{D}_{1,2}\,\frac{\partial^2 C}{\partial y^2};\\
&\rho\,\frac{\mathrm{d}H}{\mathrm{d}t}=\frac{\mu}{\sigma}\,\frac{\partial}{\partial y}\left[\frac{\partial H}{\partial y}\right.\\
&\quad+(\sigma-1)\,\frac{\partial}{\partial y}\left(\frac{u^2}{2}\right)\\
&\quad\left.+\left(\frac{1}{Le}-1\right)(h_a-h_b)\,\frac{\partial C}{\partial y}\right].
\end{aligned}\right\}\quad(1.1)$$

and the equation of thermal conductivity in a solid:

$$\rho_1 c_1\,\frac{\partial T_1}{\partial t}=\lambda_1\,\frac{\partial^2 T}{\partial y^2}.\qquad(1.2)$$

In the continuity equation, $n = 1$ for a plane case, $n = 2$ for an axisymmetrical case.

Taking into account that in the vicinity of the critical point the Mach number is small, and assuming that the Lewis number is equal to unity $Le = 1$, the energy equation becomes simplified:

$$\rho\,\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{\mu}{\sigma}\,\frac{\partial^2 h}{\partial y^2}.\qquad(1.3)$$

The following system of boundary conditions corresponds to equations (1.1), (1.2) and (1.3):

(a) on the external side of the boundary layer:

$$y\to\infty,\quad u=u_\infty=\beta x,\quad h=h_\infty,\quad C=0;$$

(b) on the sublimation wave:

$$\left.\begin{aligned}
&y=\tilde{\mathscr{D}}t,\quad u=0,\quad \rho(\tilde{\mathscr{D}}-v)=\rho_1\tilde{\mathscr{D}},\\
&(\tilde{\mathscr{D}}-v)(1-C)=\mathscr{D}_{1,2}\,\frac{\partial C}{\partial y},\\
&T=T_1,\quad \rho(\tilde{\mathscr{D}}-v)\,I=\lambda_1\,\frac{\partial T_1}{\partial y}-\lambda\,\frac{\partial T}{\partial y},\\
&C=\left[1+(\mathrm{e}^{\varepsilon/R[(1/T_0)-(1/T_B)]}-1)\,\frac{m_b}{m_a}\right]^{-1}.
\end{aligned}\right\}\quad(1.4)$$

(the latter relation is the curve of vapour pressure on the sublimation wave);

(c) at infinity in the solid:

$$y\to-\infty,\quad T=T_{-\infty}.$$

Since the stationary sublimation regime is determined below, the initial conditions are not presented.

The second last condition on the sublimation wave may be transformed. The use of both the obvious relation:

$$\lambda\,\frac{\partial T}{\partial y}=\frac{\lambda}{c_p}\,\frac{\partial h}{\partial y}-\frac{\lambda}{c_p}(h_a-h_b)\,\frac{\partial C}{\partial y}$$

and the condition of concentration conservation on the sublimation wave gives:

$$\rho(\tilde{\mathscr{D}}-v)\,[I-(h_a-h_b)(1-C)]=\lambda_1\,\frac{\partial T_1}{\partial y}-\frac{\lambda}{c_p}\,\frac{\partial h}{\partial y}.\qquad(1.5)$$

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Let us transform a system of equations in partial derivatives (1.1), (1.2) to the system of ordinary differential equations, and introduce an independent variable:

$$\eta = \sqrt{\left(\frac{u_\infty}{\nu_x}\right)}\,(y - \tilde{\mathscr{D}}t) = \sqrt{\left(\frac{\beta}{\nu}\right)}\,(y - \tilde{\mathscr{D}}t). \quad (1.6)$$

The solution of the problem is sought for in the form:

$$\left.\begin{aligned} &u = \beta x f'(\eta);\ v = -n\sqrt{(\nu\beta)}f(\eta) + \tilde{\mathscr{D}};\\ &h = h_{-\infty} + (h_\infty - h_{-\infty})\,\theta(\eta);\\ &T_1 = T_{-\infty} + (T_\infty - T_{-\infty})\,\theta_1(\eta). \end{aligned}\right\} (1.7)$$

With the aid of the transformations, equations (1.6) and (1.7), the initial system of equations is reduced as follows:

$$\left.\begin{aligned} &f'^2 - nff'' = 1 + f''',\\ &-nf\frac{dC}{d\eta} = \frac{1}{\sigma}\frac{d^2C}{d\eta^2},\\ &-nf\frac{d\theta}{d\eta} = \frac{1}{\sigma}\frac{d^2\theta}{d\eta^2},\\ &-\mathscr{D}\frac{d\theta_1}{d\eta} = \frac{1}{\sigma_1}\frac{d^2\theta}{d\eta^2}, \end{aligned}\right\} (1.8)$$

where

$$\mathscr{D} = \frac{\tilde{\mathscr{D}}}{\sqrt{\beta\nu}}, \quad \sigma_1 = \sigma\,\frac{c_1}{c_p}\cdot\frac{\rho_1}{\rho}\cdot\frac{\lambda}{\lambda_1}.$$

The boundary conditions in the variables of equations (1.6) and (1.7) can be written:

$$\eta \to \infty, \quad f'(\infty) = 1, \quad \theta = 1, \quad C = 0,$$

$$\left.\begin{aligned} &\eta = 0, \quad f'(0) = 0, \quad nf(0) = \frac{\rho_1}{\rho}\mathscr{D},\\ &nf(0)\,(1 - C)\,\sigma = \frac{dC}{d\eta},\\ &T = T_1,\ n\sigma f(0)\,\frac{I - (h_a - h_b)\,(1 - C)}{h_\infty - h_{-\infty}}\\ &\quad = \frac{\lambda_1}{\lambda}\frac{d\theta_1}{d\eta} - \frac{d\theta}{d\eta},\\ &C = \left[1 + (e^{e/R[(1/T_0) - (1/T_B)]} - 1)\,\frac{m_b}{m_a}\right]^{-1},\\ &\eta \to -\infty, \quad \theta_1 = 0. \end{aligned}\right\} (1.9)$$

2. DETERMINATION OF CONCENTRATION AND TEMPERATURE ON THE SUBLIMATION WAVE

From the heat conductivity equation for a solid it is easy to obtain the relation:

$$\frac{d\theta}{d\eta} = -\sigma_1\mathscr{D}\theta_1(0). \quad (2.1)$$

Equation (2.1) is used for transforming the condition of energy balance on the sublimation wave. After simple transformations we obtain:

$$-\frac{d\theta}{d\eta} = \sigma f(0)n\,\frac{I - (h_a - h_b)\,(1 - C_0) + c_1(T_0 - T_{-\infty})}{h_\infty - h_{-\infty}}. \quad (2.2)$$

The boundary problem, equations (1.8) and (1.9), is then reduced to the following:

$$\left.\begin{aligned} &f' - nff'' = 1 + f''';\\ &-nf\frac{dC}{d\eta} = \frac{1}{\sigma}\frac{d^2C}{d\eta^2};\\ &-nf\frac{d\theta}{d\eta} = \frac{1}{\sigma}\frac{d^2\theta}{d\eta^2} \end{aligned}\right\} (2.3)$$

under the boundary conditions

$$\left.\begin{aligned} &\eta \to \infty, \quad f'(\infty) = 1, \quad \theta = 1, \quad C = 0,\\ &\eta = 0, \quad f'(0) = 0, \quad nf(0) = \frac{\rho_1}{\rho}\mathscr{D},\\ &nf(0)\,(1 - C)\,\sigma = \frac{dC}{d\eta},\\ &-\frac{d\theta}{d\eta} = n\sigma f(0) \times\\ &\frac{I - (h_a - h_b)\,(1 - C_0) + c_1(T_0 - T_{-\infty})}{h_\infty - h_{-\infty}},\\ &C = \left[1 + (e^{e/R[(1/T_0) - (1/T_B)]} - 1)\,\frac{m_b}{m_a}\right]^{-1}. \end{aligned}\right\} (2.4)$$

It is easy to see that the second and third equations may be easily integrated, function $f(\eta)$ still remaining unknown.

Upon integration of the equations mentioned and determination of the constants of the boundary conditions, the expression for concentration and heat content distribution in a gas mixture is obtained:

$$\left.\begin{aligned} C(\eta) &= n\sigma f(0)\,\frac{\phi(\eta)-\phi(\infty)}{1-\phi(\infty)\sigma f(0)n};\\ \theta(\eta) &= n\sigma f(0)\\ &\times\frac{l-(h_a-h_b)(1-C_0)+c_1(T_0-T_{-\infty})}{h_\infty-h_{-\infty}}\\ &\times[\phi(\infty)-\phi(\eta)]+1, \end{aligned}\right\}(2.5)$$

where

$$\phi(\eta)=\int_0^\eta e^{-\sigma n\int_0^t f(t)\,dt}\,.\,dt.$$

From equation (2.5) the expression for concentration and heat content on the sublimation wave may be easily obtained:

$$\left.\begin{aligned} C_0 &= -\frac{n\sigma f(0)\phi(\infty)}{1-n\sigma f(0)\phi(\infty)};\\ \theta_0 &= n\sigma f(0)\phi(\infty)\\ &\times\frac{l-(h_a-h_b)(1-C_0)+c_1(T_0-T_{-\infty})}{h_\infty-h_{-\infty}}. \end{aligned}\right\}(2.6)$$

Eliminating from equation (2.5) the combination of values $n\sigma f(0)\phi(\infty)$, where $f(0)$ and $\phi(\infty)$ are still unknown, the finite relation connecting concentration and heat values on the sublimation wave is obtained:

$$\theta(0)=\frac{h_0-h_{(-\infty)}}{h_\infty-h_{(-\infty)}}=1+\frac{C_0}{C_0-1}\times\frac{l-(h_a-h_b)(1-C_0)+c_1(T_0-T_{-\infty})}{h_\infty-h_{-\infty}}. \quad (2.7)$$

The heat content of a mixture may be expressed through the heat contents of components:

$$h_0=C_0h_a+(1-C_0)h_b, \quad (2.8)$$

then equation (2.7) is reduced to the form:

$$\frac{l+c_1(T_0-T_{-\infty})}{h_b(0)-h(\infty)}=\frac{C_0-1}{C_0}. \quad (2.9)$$

Adding to equation (2.9) the equation of the vapour pressure curve on the sublimation wave, the system of two finite equations with two unknowns, which is easily solved, may be obtained.

Thus, to obtain temperature T_0 and concentration C_0 on the sublimation wave it is necessary to solve the system of equations:

$$\frac{l+c_1(T_0-T_{-\infty})}{h_b(0)-h_\infty}=\frac{C_0-1}{C_0};$$

$$C_0=\left[1+(e^{e/R[(1/T_0)-(1/T_B)]}-1)\,\frac{m_b}{m_a}\right]^{-1}. \quad (2.10)$$

3. DETERMINATION OF VELOCITY OF SUBLIMATION FRONT PROPAGATION

In addition to concentration and heat content values on the sublimation wave which are, in fact, determined, it is necessary to know the velocity of propagation of the sublimation front.

From the first equation (2.6) it is easy to obtain:

$$\frac{C_0}{C_0-1}=n\sigma f(0)\phi(\infty)=n\sigma f(0)\int_0^\infty e^{-n\sigma\int_0^t f(t)\,dt}\,.\,dt. \quad (3.1)$$

If the dependence of $\phi(\infty)$ upon $f(0)$ were known, then equation (3.1) would give the relation between concentration on the sublimation wave and the dimensionless velocity of propagation of the sublimation front $f(0)$. Then, by the known C_0 it would be easy to find $f(0)$, and the problem would be completely solved.

Thus, the equation of motion should be considered, to find the dependence of $\phi(\infty)$ upon $f(0)$:

$$f'^2-nff''=f'''+1 \quad (3.2)$$

with the boundary conditions

$$f(0)=a,\quad f'(0)=0,\quad f'(\infty)=1.$$

Solving numerically the boundary problem, equation (3.2), we obtain the dependence of $f''(0)$ upon $f(0)=a$, the form of which for plane and axisymmetrical cases is presented in Fig. 2.

Knowing the dependence of $f''(0)$ upon $f(0)$, we get the dependence of the function $f(\eta)$ upon $f(0)$ in the form of the Maclaurin series:

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$$f(\eta) = a + \frac{a}{2}\eta^2 - \frac{n\alpha a + 1}{3!}\eta^3 + \frac{n\alpha(n\alpha a + 1)}{4!}\eta^4 + \frac{(2-n)a^2 - n^2\alpha^2(n\alpha a + 1)}{5!}\eta^5$$
$$+ \frac{(-8+5n)n\alpha a^2 + (-6+4n+n^4\alpha^4)a + n^3\alpha^3}{6!}\eta^6$$
$$+ \frac{(22-16n)n^2\alpha^2 a^2 + (26-19n-n^4\alpha^4)n\alpha a + (6-4n-n^4\alpha^4)}{7!}\eta^7 + \ldots, \qquad (3.3)$$

where $$a = f(0); \quad a = f''(0).$$

To find the dependence of $J = n\sigma f(0)\phi(\infty)$ upon $a = f(0)$, we evaluate the integral $\phi(\infty)$ for a set of values a:

$$\phi_\infty = \int_0^\infty e^{-n\sigma\int_0^t f(t)dt} \cdot dt, \qquad (3.4)$$

applying the asymptotic method [2].

On integrating $f(\eta)$ and substituting the result into equation (3.4) we obtain:

$$\phi(\infty) = \int_0^\infty e^{-n\sigma a\eta - n\sigma z(\eta)} \cdot d\eta, \qquad (3.5)$$

where

$$z(\eta) = \frac{a}{3!}\eta^3 - \frac{n\alpha a + 1}{4!}\eta^4 + \frac{n\alpha(n\alpha a + 1)}{5!}\eta^5 + \frac{(2-n)a^2 - n^2\alpha^2(n\alpha a + 1)}{6!}\eta^6$$
$$+ \frac{(-8+5n)n\alpha a^2 + (-6+4n+n^4\alpha^4)a + n^3\alpha^3}{7!}\eta^7$$
$$+ \frac{(22-16n)n^2\alpha^2 a^2 + (26-19n-n^4\alpha^4)n\alpha a + (6-4n-n^4\alpha^4)}{8!}\eta^8 + \ldots \qquad (3.6)$$

Changing the variable of integration from η to z, the limits of integration not being changed:

$$\phi(\infty) = \int_0^\infty e^{-n\sigma z} e^{-n\sigma a\eta} \cdot \frac{d\eta}{dz} \cdot dz. \qquad (3.7)$$

The integrand contains an unknown function of z:

$$e^{-n\sigma a\eta} \cdot \frac{d\eta}{dz},$$

which will be sought for in the form of a series:

$$e^{-n\sigma a\eta} \cdot \frac{d\eta}{dz} = z^{-2/3} \sum_{m=0}^{\infty} d_m z^{m/3} \qquad (3.8)$$

The expansion coefficients d_m are determined using the known formula:

$$d_m = \frac{1}{6\pi_i}\phi \cdot \exp(-an\sigma\eta) z^{-(m+1)/3} \cdot d\eta \qquad (3.9)$$

From equation (3.9) it follows that d_m is $\frac{1}{3}$ of the coefficient of η^{-1} in the expansion of the function $\exp(-n\sigma a\eta)\ z^{-(m+1)/3}$ in integral powers of η.

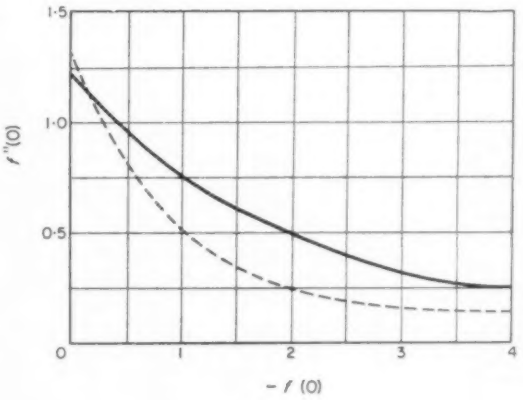

FIG. 2.
——— plane case;
— — — axisymmetrical case.

The expansion of $\exp(-n\sigma a\eta)\, z^{-(m+1)/3}$ as a series is as follows:

$$\exp(-n\sigma a\eta) = \sum_{m=0}^{\infty} (-1)^m \frac{(n\sigma a\eta)^m}{m!}. \qquad (3.10)$$

The expansion of $z^{-(m+1)/3}$ as a series in integral powers of η is obtained. For this, equation (3.6) is raised to the power $\frac{1}{3}$, and the expression obtained is expanded as a power series:

$$z^{-(m+1)/3} = \eta^{-(m+1)} (C_0 + C_1\eta + C_2\eta^2 + C_3\eta^3 + C_4\eta^4 + C_5\eta^5 + \ldots), \qquad (3.11)$$

where

$$C_0 = \left(\frac{a}{3!}\right)^{-(m+1)/3};$$

$$C_1 = \frac{m+1}{3}\left(\frac{a}{3!}\right)^{-(m+4)/3} \frac{n\sigma a + 1}{4!};$$

$$C_2 = \frac{1}{2}\,\frac{m+1}{3}\left(\frac{a}{3!}\right)^{-(m+4)/3}\left\{\frac{m+4}{3}\left(\frac{a}{3!}\right)^{-1}\left(\frac{n\sigma a+1}{4!}\right)^2 - \frac{2}{5!}\,n\sigma\,(n\sigma a+1)\right\};$$

$$C_3 = \frac{1}{3!}\,\frac{m+1}{3}\left(\frac{a}{3!}\right)^{-(m+4)/3}\left\{\frac{(m+4)(m+7)}{3^2}\left(\frac{a}{3!}\right)^{-2}\left(\frac{n\sigma a+1}{4!}\right)^3\right.$$
$$\left. - (m+4)\left(\frac{a}{3!}\right)^{-1}\frac{(n\sigma a+1)}{4!}\,\frac{2}{5!}\,n\sigma\,(n\sigma a+1) - \frac{3!}{6!}\,[(2-n)a^2 - n^3\sigma^3 a - n^2\sigma^2]\right\};$$

$$C_4 = \frac{1}{4!}\,\frac{m+1}{3}\left(\frac{a}{3!}\right)^{-(m+4)/3}\left\{\frac{(m+4)(m+7)(m+10)}{3^3}\left(\frac{a}{3!}\right)^{-3}\left(\frac{n\sigma a+1}{4!}\right)^4\right.$$
$$- 6\,\frac{(m+4)(m+7)}{3^2}\left(\frac{a}{3!}\right)^{-2}\left(\frac{n\sigma a+1}{4!}\right)^2\frac{2}{5!}\,n\sigma\,(n\sigma a+1) + (m+4)\left(\frac{a}{3!}\right)^{-1}\left[\frac{2}{5!}\,n\sigma\,(n\sigma a+1)\right]^2$$
$$- 4\,\frac{m+4}{3}\left(\frac{a}{3!}\right)^{-1}\frac{n\sigma a+1}{4!}\,\frac{3!}{6!}\,[(2-n)a^2 - n^3\sigma^3 a - n^2\sigma^2]$$
$$\left. - \frac{4!}{7!}\,[(-8+5n)n\sigma a^2 + (-6+4n+n^4\sigma^4)a + n^3\sigma^3]\right\};$$

$$C_5 = \frac{1}{5!}\,\frac{m+1}{3}\left(\frac{a}{3!}\right)^{-(m+4)/3}\left\{\frac{(m+4)(m+7)(m+10)(m+13)}{3^4}\left(\frac{a}{3!}\right)^{-4}\left(\frac{n\sigma a+1}{4!}\right)^5\right.$$
$$- 10\,\frac{(m+4)(m+7)(m+10)}{3^3}\left(\frac{a}{3!}\right)^{-3}\left(\frac{n\sigma a+1}{4!}\right)^3\cdot\frac{2}{5!}\,n\sigma(n\sigma a+1)$$
$$+ 15\,\frac{(m+4)(m+7)}{3^2}\left(\frac{a}{3!}\right)^{-2}\left(\frac{n\sigma a+1}{4!}\right)\left[\frac{2}{5!}\,n\sigma(n\sigma a+1)\right]^2 - 10\,\frac{(m+4)(m+7)}{3^2}\left(\frac{a}{3!}\right)^{-2}$$
$$\times\left(\frac{n\sigma a+1}{4!}\right)^2\frac{3!}{6!}\,[(2-n)a^2 - n^3\sigma^3 a - n^2\sigma^2] + 10\,\frac{(m+4)}{3}\left(\frac{a}{3!}\right)\frac{2}{5!}\,n\sigma(n\sigma a+1)\,\frac{3!}{6!}$$
$$\times[(2-n)a^2 - n^3\sigma^3 a - n^2\sigma^2] - 5\,\frac{m+4}{3}\left(\frac{a}{3!}\right)^{-1}\frac{n\sigma a+1}{4!}\,\frac{4!}{7!}\,[(-8+5n)\,n\sigma a^2$$
$$+ (-6+4n+n^4\sigma^4)a + n^3\sigma^3 - \frac{5!}{8!}\,[(22-16n)n^2\sigma^2 a^2 + (26-19n-n^4\sigma^4)n\sigma a$$
$$\left. + (6-4n-n^4\sigma^4)]\right\}.$$

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Multiplying the right terms of equations (3.10) and (3.11), the expansion of

$$\exp(-n\sigma a\eta) \cdot z^{-(m+1)/3}$$

as a series may be obtained.

In the expansion obtained, $\frac{1}{3}$ of the coefficient of η^{-1} is equal to the coefficient a_m. Thus, coefficients d_m for a set of values $a = f(0)$ are calculated.

Using z as the variable of integration, the integral $\phi(\infty)$ has the form:

$$\phi(\infty) = \int_0^\infty (d_0 + d_1 z^{1/3} + d_2 z^{2/3} + d_3 z + \ldots) z^{-2/3} \cdot e^{-n\sigma z} \cdot dz. \quad (3.12)$$

On integrating we obtain:

$$\phi(\infty) = n\sigma f(0) \sum_{m=0}^{\infty} d_m (n\sigma)^{-(m+1)/3} \cdot \Gamma\left(\frac{m+1}{3}\right) \quad (3.13)$$

where Γ is the gamma-function.

At the known coefficients d_m, the integral $\phi(\infty)$ may be easily calculated for a set of values $a = f(0)$.

Hence, the dependence of the integral $\phi(\infty)$ upon $a = f(0)$ is found. Then, from the equality, (3.1), the dependence of $C_0/(C_0 - 1)$ upon $a = f(0)$ becomes known. This dependence is presented in Fig. 3. The curves are plotted for plane and axisymmetrical cases at Prandtl numbers 0·7 and 1.

Since the problem of determining the concentration C_0 on the sublimation wave is already solved (section 2), the dimensionless velocity of propagation of the sublimation front $f(0)$ is determined graphically from Fig. 2.

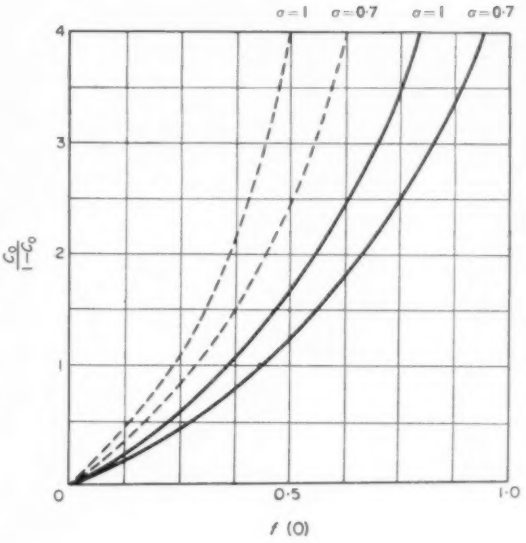

FIG. 3.
— — — — plane case;
———— axisymmetrical case.

The dimensional velocity of propagation of the sublimation front is expressed through $f(0)$ as follows:

$$\tilde{\mathcal{D}} = \frac{\rho}{\rho_1} \sqrt{(\beta\nu)}\, nf(0) \quad (3.14)$$

REFERENCES

1. LEONARD ROBERTS, *NASA, TR* (1959).
2. D. MEKSYN, *Proc. Roy. Soc.* A **192**, 567 (1948).

Abstract—The paper gives an exact solution of a solid sublimation problem near a critical point in flat and axisymmetrical flows of an incompressible gas. Finite relations are established for determining concentration and temperature on a wave of sublimation. Plots for the velocity determination of sublimation front propagation are given.

An analogous problem is considered by Roberts [1]. However, he gives an approximate solution using the integral Pohlhausen method (the distribution of functions in a boundary layer is given).

Résumé—Les auteurs présentent une solution exacte du problème de la sublimation d'un solide au voisinage d'un point critique dans des écoulements plans et de révolution de gaz incompressible. Ils ont établi des relations finies permettant de déterminer la concentration et la température sur une onde de sublimation et donnent des courbes de la vitesse de propagation du front de sublimation.

Un problème analogue a été étudié par Roberts [1], mais ce dernier donne une solution approchée utilisant la méthode intégrale de Pohlhausen (la distribution des fonctions dans une couche limite est donnée).

Zusammenfassung—Für die Sublimation in der Nähe des kritischen Punktes im ebenen und achsensymmetrischen Strom eines nichtkompressiblen Gases wurde eine exakte Lösung ermittelt. Konzentration und Temperatur im Sublimationsbereich können nach angegebenen Beziehungen bestimmt werden. Aus Diagrammen ist die Geschwindigkeit, mit der die Sublimationsfront fortschreitet, zu entnehmen.

Ein analoges Problem ist von Roberts [1] bearbeitet worden. Mit Hilfe der integralen Pohlhausen-Methode erhält er eine Näherungslösung (bei gegebener Funktionsverteilung in der Grenzschicht).

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A METHOD OF SOLUTION OF THE HEAT CONDUCTION (DIFFUSION) EQUATION

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Аннотация—Разработан метод решения уравнения теплопроводности (диффузии) для нестационарной трехмерной задачи при наличии конвекции. Дано теоретическое обоснование метода для случая, когда горизонтальная (продольная) составляющая скорости перемещения среды суть произвольный полином относительно вертикальной координаты. Решение уравнения базируется на операционном методе и имеет вид ряда, расположенного по степеням обобщенного параметра Лапласа.

NOMENCLATURE

T, temperature;
x, y, horizontal plane;
z, vertical co-ordinate;
$u(z) = \sum_{m=0} a_m z^m$ and
W, corresponding velocities of medium displacement;
λ, λ_z, corresponding coefficients of heat conduction;
T_0, Green's function.

1. METHOD OF SOLVING EQUATION (1.1)

THE heat equation:

$$\frac{\partial T}{\partial t} + W\frac{\partial T}{\partial z} + u(z)\frac{\partial T}{\partial x} = \lambda\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \lambda_z \frac{\partial^2 T}{\partial z^2} \qquad (1.1)$$

is solved for the following initial and boundary conditions, assuming that W, λ and λ_z are constant:

at

$$t = 0, \quad T = Q\delta(x)\,\delta(y)\,\delta(z-h);$$

at

$$\sqrt{(x^2+y^2+z^2)} \to \infty \quad T = 0;$$

$$\left(\lambda_z \frac{\partial T}{\partial z} + WT\right)_{z=0} = 0.$$

The method of solution is illustrated in application to the equation:

$$\frac{\partial T}{\partial t} + W\frac{\partial T}{\partial z} + az\frac{\partial T}{\partial x} = \lambda\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \lambda_z \frac{\partial^2 T}{\partial z^2}. \qquad (1.2)$$

Equation (1.2) is solved instead of (1.1) only to simplify the presentation, and to avoid cumbersome computations which hinder an understanding of the method.

In future it will not be essential that a horizontal velocity component of medium displacement be given as a linear function of the co-ordinate z. The method proposed allows for solution when the horizontal velocity component of medium displacement is given as polynomials of any whole powers with respect to z.

Substituting the variables in equation (1.2):

$$\xi = \frac{x}{\sqrt{\lambda}}, \quad \eta = \frac{y}{\sqrt{\lambda}}, \quad \zeta = \frac{z}{\sqrt{\lambda_z}}, \quad W = \frac{W}{\sqrt{\lambda_t}},$$

we obtain:

$$\frac{\partial T}{\partial t} + W\frac{\partial T}{\partial \zeta} + b\zeta\frac{\partial T}{\partial \xi} = \frac{\partial^2 T}{\partial \xi^2} + \frac{\partial^2 T}{\partial \eta^2} + \frac{\partial^2 T}{\partial \zeta^2}, \quad (1.3)$$

where

$$b = a\sqrt{\left(\frac{\lambda_z}{\lambda}\right)}.$$

Equation (1.3) is solved by the operational method. With the help of the Laplace triple transformation:

$$T^* = \int_0^\infty T\,e^{-P_t t}\,dt \int_{-\infty}^{\infty}\int e^{-P_\xi \xi - P_\eta \eta}\,d\xi\,d\eta$$

substituting

$$T^* = \chi\, e^{(W/2)\zeta}$$

equation (1.3) is reduced to the form:

$$\frac{d^2\chi}{d\zeta^2} - P\chi = bP_\xi \zeta\chi - \frac{Q}{\lambda}\, e^{-(W/2)\zeta}\,\delta[\zeta\sqrt{(\lambda_t)} - h], \quad (1.4)$$

where

$$P = P_t - P_\xi^2 - P_\eta^2 - \frac{W^2}{4}.$$

Assuming that $\zeta\sqrt{(P)} = \zeta^*$, we reduce equation (1.4) to the form:

$$\frac{d^2\chi}{d\zeta^{*2}} - [1 + 2\zeta^*]\chi = \frac{Q}{P\lambda}\, e^{-W\zeta^*/2\sqrt{(P)}}\,\delta\left(\frac{\lambda_z}{P}\,\zeta^* - h\right), \quad (1.5)$$

where

$$\frac{bP_\xi}{P^{3/2}}.$$

Let us solve the homogeneous equation:

$$\frac{d^2\chi}{d\zeta^{*2}} - [1 + a\zeta^*]\chi = 0. \quad (1.6)$$

We shall find the solution of this equation in the form of the following series arranged according to the powers of a:

$$\chi = \chi_0 + a\chi_1 + a^2\chi_2 + \ldots + a^n\chi_n + \ldots \quad (1.7)$$

Substituting equation (1.7) into (1.6) and equating the terms of equal powers of a, we get the following infinite system of ordinary differential equations:

$$\left.\begin{aligned} &\frac{d^2\chi_0}{d\zeta^{*2}} - \chi_0 = 0,\\ &\frac{d^2\chi_1}{d\zeta^{*2}} - \chi_1 = \zeta^*\chi_0,\\ &\frac{d^2\chi_n}{d\zeta^{*2}} - \chi_n = \zeta^*\chi_{n-1}. \end{aligned}\right\} \quad (1.8)$$

The first equation of system (1.8) has two fundamental solutions:

$$e^{\zeta^*} \text{ and } e^{-\zeta^*}.$$

The two fundamental solutions for equation (1.5) may be obtained with the help of the generating operator:

$$\chi_n = \tfrac{1}{2}\,[e^{\zeta^*}\int e^{-\zeta^*}\varphi_{n-1}(\zeta^*)\,d\zeta^* - e^{-\zeta^*}\int e^{\zeta^*}\varphi_{n-1}(\zeta^*)d\zeta^*], \quad (1.9)$$

where

$$\varphi_{n-1}(\zeta^*) = \zeta^*\chi_{n-1}(\zeta^*).$$

Assuming first $\chi_0 = e^{\zeta^*}$ we get the first fundamental solution of equation (1.6) in the form: $\chi_1 = e^{\zeta^*}\,\mathscr{F}_1(\zeta^*)$. The second fundamental solution of (1.6) may be obtained in an analogous way:

$$\chi_2 = e^{-\zeta^*}\,\mathscr{F}_2(\zeta^*).$$

The general solution of equation (1.5) for the boundary conditions given above will be written as follows [1]:

$$\left.\begin{aligned} \chi = {} & \frac{Q\,e^{-(Wh/2\lambda_z)}}{P\lambda\sqrt{\lambda_z}}\,\frac{1}{W}\left[\chi_1(h,P) + \frac{(W/2)\chi_1(0) - \sqrt{(P\lambda_z)}\,\chi_1'(0)}{-(W/2)\chi_2(0) + \sqrt{(P\lambda_z)}\,\chi_2'(0)}\,\chi_2(h,P)\right]\\ & \times \chi_2(\zeta^*,P), \quad \zeta^* \geqslant h\sqrt{\left(\frac{P}{k_z}\right)}\ (z \geqslant h); \end{aligned}\right\} \quad (1.10)$$

$$\left.\begin{aligned} \chi = {} & \frac{Q\,e^{-(Wh/2\lambda_t)}}{P\lambda\sqrt{\lambda_t}}\,\frac{1}{W}\left[\chi_1(\zeta^*,P) + \frac{(W/2)\chi_1(0) - \sqrt{(P\lambda_z)}\,\chi_1'(0)}{-(W/2)\chi_2(0) + \sqrt{(P\lambda_z)}\,\chi_2'(0)}\right.\\ & \left.\times\,\chi_2(\zeta^*,P)\right]\chi_2(h,P),\\ & \zeta^* \leqslant h\sqrt{\left(\frac{P}{k_z}\right)}\ (z \leqslant h); \end{aligned}\right\} \quad (1.11)$$

where W is the Wronsky determinant.

$$\begin{vmatrix} \chi_1 & \chi_2 \\ \chi_1' & \chi_2' \end{vmatrix}$$

Expressions (1.10) and (1.11) represent the image of the function T.

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The transition from the image to the original is carried out with the help of the formulae:

$$L^{-1}\left(\frac{1}{\lambda^{(n/2)+1}}\,e^{-a\sqrt{(\lambda)}}\right) = (4t)^{n/2}\,i^n \operatorname{erf} c\,\frac{a}{2\sqrt{t}} \quad (1.12)$$

and

$$L_{\xi,\eta}^{-1}\{P_\xi^n \exp[-(P_\xi^2+P_\eta^2)t]\} = \frac{(-1)^n}{4\pi t}\left(\frac{1}{2\sqrt{t}}\right)^n \exp[-(y^2/4\lambda t)-\omega^2] \quad (1.13)$$

Here $i^n \operatorname{erf} c\,x$ is the integral function of errors of the n-th order; H_n are Hermite's polynomials [2, 3]; $\omega = \xi/2\sqrt{t}$.

The solution for equation (1.13) has the following form:

$$T = \mathscr{F}\left[i^n \operatorname{erf} c\,\frac{z}{2\sqrt{(\lambda_z t)}},\ H_n\left(\frac{x}{2\sqrt{(k_x t)}}\right) y, z, t\right]. \quad (1.14)$$

Solution (1.14) obtained for an instantaneous point source is Green's function for equation (1.3), with more general assumptions on the time and space characteristics of an admixture source.

With the solution of equation (1.3) we may obtain the solution of any non-homogeneous problem corresponding to this equation. Indeed, letting equation (1.3) have the term $f(x, y, z, t)$ in the right-hand side, the solution satisfying non-homogeneous initial and boundary conditions:

$$T(x, y, z, 0) = \varphi(x, y, z),$$

$$\left[\lambda_z \frac{\partial T}{\partial z} + WT\right]_{z=0} = \psi(x, y, t),$$

at

$$\sqrt{(x^2+y^2+t^2)} \to \infty \quad T = 0,$$

will have the form:

$$\left.\begin{aligned} T(x, y, z, t) = {} & \frac{1}{Q}\int_0^t d\tau \int_{-\infty}^{\infty}\int d\xi\, d\eta \\ & \times \int_0^{\infty} f(x-\xi, y-\eta, \zeta, t-\tau)T_0 \\ & \times (\xi, \eta, z, \tau, \zeta)\, d\zeta + \frac{1}{Q}\int_{-\infty}^{\infty}\int d\xi\, d\eta \\ & \times \int_0^{\infty} \phi(x-\xi, y-\eta, \zeta)T_0 \\ & \times (\xi, \eta, z, t, \zeta)\, d\zeta - \frac{1}{Q}\int_0^t d\tau \\ & \times \int_{-\infty}^{\infty}\int \psi(x-\xi, y-\eta, t-\tau)T_0 \\ & \times (\xi, \eta, z, \tau; 0)\, d\zeta\, d\eta). \end{aligned}\right\} \quad (1.15)$$

At the boundary condition:

$$T|_{z=0} = \psi_1(x, y, t)$$

the last term in (1.15) should be substituted by the expression

$$\frac{1}{Q}\int_0^t d\tau \int_{-\infty}^{\infty}\int \psi_1(x-\xi, y-\eta, t-\tau) \times \left[\lambda_z \frac{\partial T_0(\xi, \eta, z, t; \zeta)}{\partial \zeta}\right]_{\zeta=0} d\xi\, d\eta.$$

Let a stationary source with intensity q of heat units per time unit be concentrated at the point $x = y = 0$, $z = h$. The temperature distribution may be obtained as a limit of solution for equation (1.3) at $t \to \infty$, with the right-hand side

$$f(x, y, z, t) = q\delta(x)\,\delta(y)\,\delta(z-h)$$

at homogeneous initial and boundary conditions.

The first term of equation (1.15) shows that

$$T_1(x, y, z; h) = \frac{q}{Q}\int_0^{\infty} T_0(x, y, z, t; h)\, dt.$$

2. THEORETICAL SUBSTANTIATION OF THE METHOD

To substantiate the method proposed above, the following theorems were proved:

(1) Let $\prod_{k=0}^{n} a_k \sim n^{a[4]}$ (*) at sufficiently large n and $a_0 \leqslant a_1 \leqslant \ldots \leqslant a_n \ldots$ or $a_0 > a_1 \ldots > a_n > \ldots$. Then, at sufficiently large n,

$$\prod_{k=0}^{n} a_{km} \sim n^{a/m}\, u.$$

Therefore:

$$\prod_{m=2}^{n}\left(1+\frac{1}{(m+1)k+m-1}\right)\sim n^{1/k+1}. \quad (2.1)$$

We used the symbol $\sim$ with the meaning $a_n \sim n^\alpha$ when $n\to\infty$, if the condition $a_n = 0(n^\beta)$ is fulfilled when $\beta \leqslant \alpha$.

(2) Let the function $\mathscr{F}_1(\xi, z, t)$ be the original of the image $P_\xi^n f(P, z)$, and the function $\mathscr{F}_2(\xi, z, t)$, that of $P_\xi^{n+1} f(P, z)$. Then for a sufficiently large number n, excluding the neighbourhood of zeros of the function $\mathscr{F}_2(\xi, z, t)$,

$$\left|\frac{\mathscr{F}_2(\xi, z, t)}{\mathscr{F}_1(\xi, z, t)}\right| \sim \sqrt{\frac{n}{t}}. \quad (2.2)$$

(3)

$$\max\frac{i^n \operatorname{erf} c\,x}{i^{n-1}\operatorname{erf} c\,x} = \frac{i^n \operatorname{erf} c\,0}{i^{n-1}\operatorname{erf} c\,0} \sim \frac{1}{\sqrt{n}}. \quad (2.3)$$

The main idea of the proof that there exists the original for the general solution of the equation when $u(z) = \sum_{\nu=0}^{k} r_\nu \zeta^\nu$ is the following [1]:

We first prove the existence of the original for the function χ which is the solution for the equation:

$$\frac{d^2\chi}{d\zeta^2} - P\chi = P_\xi u(\zeta)\chi$$

or

$$\frac{d^2\chi}{d\zeta^{*2}} - \chi = \frac{1}{P} P_\xi u(\zeta^*)\chi. \quad (2.4)$$

The formal solution of equation (2.4) may be obtained in the form:

$$\chi = \sum_{n=0}^{\infty} \alpha^n \chi_n,$$

where $\alpha = f(P_\xi, P, r_\nu)$ is the definite parameter chosen.

The succession of the functions χ_n is determined with the help of the operator:

$$\left.\begin{aligned} &e^{-\zeta^*} A_{R_k}(\chi_n) \\ &= \tfrac{1}{2}[e^{\zeta^*}\int e^{-2\zeta^*} R_k(\zeta^*)\,\chi_n(\zeta^*)\,d\zeta^* \\ &- e^{-\zeta^*}\int R_k(\zeta^*)\chi_n(\zeta^*)\,d\zeta^*], \end{aligned}\right\} \quad (2.5)$$

so

$$\chi_{n+1} = A_{R_k}(\chi_n);\quad \chi_0 = 1.$$

The proof is given here for the existence of the fundamental solution $e^{-\zeta^*}\mathscr{F}_2(\zeta^*)$.

Let us formulate the main properties of the operator A_{R_k}. Let $|q_\nu| > |P_\nu|$ for coefficients of polynomials of one order

$$Q_m = \sum_{\nu=0}^{m} q_\nu \zeta^\nu \quad \text{and} \quad P_m = \sum_{\nu=0}^{m} P_\nu \zeta^\nu$$

where q_ν are the coefficients of the polynomial Q_m of one sign. In this meaning we shall use $Q_m \gg P_m$. Besides equation (2.4) we shall consider the following equation:

$$\frac{d^2\chi^*}{d\zeta^{*2}} - \chi^* = \frac{1}{P} P_\xi S_k(\zeta^*)\chi^*, \quad (2.6)$$

where

$$S_k(\zeta) = \sum_{\nu=0}^{k} S_\nu \zeta^\nu \gg R_k(\zeta).$$

Then we can prove that $A_{S_k}(Q_m) \gg A_{R_k}(P_m)$.

Thus, if for equation (2.6) we build a formal solution in the form of the succession $\chi^*_{n+1} = A_{S_k}(\chi^*_n)$, and assume that $\chi_0 = \chi^*_0 = 1$, then it appears that:

$$\chi^*_1 = A_{S_k}(1) \gg A_{R_k}(1) = \chi_1$$

$$\chi^*_2 = A_{S_k}(\chi^*_1) \gg A_{R_k}(\chi_1) = \chi_2$$

and

$$\chi^*_n \gg \chi_n.$$

Note that by presenting S_k in the form of $S_k = a(\zeta + N)^k$, and by choosing the constants a and N correspondingly, one may attain the fulfilment of the relation $S_k \gg R_k$. By substituting the variable $2(\zeta + N)\sqrt{(P)} = \kappa$, equation (2.6) is reduced to the form:

$$\frac{d^2\chi^*}{d\kappa^2} - \tfrac{1}{4}\chi^* = \frac{aP_\xi}{2^{k+2}P^{(k/2)+1}}\kappa^k\chi^*.$$

By the method of complete induction it was shown [1] that terms of succession of the functions χ_n given in the form:

$$\bar{\chi}_n = \frac{1}{n!(k+1)^n}\left[\kappa + M(k+1)\prod_{m=2}^{n}\frac{m(k+1)}{m(k+1)-1}\right]^{n(k+1)} \quad (2.7)$$

where

$$M \geqslant 2,$$

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and of the succession of the solution for equation (2.6) are bound by the condition

$$\bar{\chi}_n \geqslant \chi_n^*. \tag{2.8}$$

Let us prove the convergence of the original $e^{-(\kappa/2)} \sum_{n=0}^{\infty} \alpha^n \bar{\chi}_n$. Apparently, on the basis of equation (2.8) the convergence of the original $e^{-(\kappa/2)} \sum_{n=0}^{\infty} \alpha^n \chi_n^*$ will be proved automatically.

The proof of the convergence is based on the estimation of the expression:

$$\frac{L_{\xi,t}^{-1}(e^{-(\kappa/2)}\,\alpha^{n+1}\,\bar{\chi}_{n+1})}{L_{\xi,t}^{-1}(e^{-(\kappa/2)}\,\alpha^{n}\,\bar{\chi}_{n})}.$$

Taking into account equations (2.1)–(2.3) it may be shown that

$$\left|\frac{L_{\xi,t}^{-1}(e^{-(\kappa/1)}\,\alpha^{n+1}\,\bar{\chi}_{n+1})}{L_{\xi,t}^{-1}(e^{-(\kappa/2)}\,\alpha^{n}\,\bar{\chi}_{n})}\right| \sim a\,\frac{[2(\zeta+N)]^{k+1}}{(n+1)(k+1)} \tag{2.9}$$

Thus, the existence of the original of one of the fundamental solutions for equation (2.4) is proved. Proceeding from equation (2.9) and (2.5) it is easy to show that the fundamental solution for equation (2.4) is $\chi_1 = e^{\zeta^*}\mathscr{F}_1(\zeta^*) \to \infty$ at $\zeta^* \to \infty$ and $\chi_2 = e^{-\zeta^*}\mathscr{F}_2(\zeta^*) \to 0$ at $\zeta^* \to \infty$.

By this it was proved that the solution of equation (1.1) tends to zero at $z \to \infty$ (see equation (1.10)).

Now we shall turn to prove the existence of the original for the image which is the general solution for equation (1.1).

Let us consider the expression [5]:

$$f(P_t) = \frac{1 + \dfrac{a_1}{P_t^{1/2}} + \dfrac{a_2}{P_t} + \ldots + \dfrac{a_n}{P_t^{n/2}} + \ldots}{1 + \dfrac{b_1}{P_t^{1/2}} + \dfrac{b_2}{P_t} + \ldots + \dfrac{b_n}{P_t^{n/2}}} \quad (*)$$

Let us expand the function $f(P_t)$ in a series by $1/P_t^{1/2}$. We then obtain:

$$f(P_t) = \Delta_0 - \Delta_1 \frac{1}{P_t^{1/2}} + \Delta_2 \frac{1}{P_t} + \ldots + (-1)^n \Delta_n \frac{1}{P_t^{n/2}} + \ldots \quad (**)$$

$$\Delta_0 = 1$$

$$\Delta_1 = \begin{vmatrix} 1 & 1 \\ a_1 & b_1 \end{vmatrix}; \quad \Delta_2 = \begin{vmatrix} 1 & 1 & 0 \\ a_1 & b_1 & 1 \\ a_2 & b_2 & b_1 \end{vmatrix}$$

$$\Delta_k = \begin{vmatrix} 1 & 1 & 0 & 0 & \ldots & 0 \\ a_1 & b_1 & 1 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{k-1} & b_{k-1} & b_{k-2} & \ldots & & 1 \\ a_k & b_k & b_{k-1} & \ldots & & b_1 \end{vmatrix}$$

The series is convergent outside any ring with a centre in the origin of co-ordinates $P_t = 0$, including all the poles of the function $f(P_t)$. The transformation of it into the language of the variable t may be carried out using the operational relation term by term:

$$\sum_{n=1}^{\infty} \frac{a_n}{P_t^{n/2}} \doteqdot U(t) \sum_{n=1}^{\infty} a_n \frac{t^{n/2}}{P[1 + (n/2)]}, \; Re\, P_t > 0$$

This follows from the fact that the sum of any convergent series always equals its Borelevsky's sum:

$$\sum_{n=1}^{\infty} c_n = \int_0^{\infty} e^{-t} \sum_{n=1}^{\infty} \frac{c_n}{(n/2)!}\, t^{n/2}\, dt.$$

Thus, the existence of the original for the image of the type $\chi_1(0)/\chi_2(0)$ is proved (see (1.10)). In an analogous way the existence of the original for any terms in expressions (1.10) and (1.11) may be proved.

Proceeding from the succession of (2.7), it is easy to prove the absolute convergence of derivatives entering equation (1).

Thus, the formal series obtained by our method absolutely converges with all its derivatives, and is in fact the solution of equation (1.1).

REFERENCES

1. O. S. BERLYAND, Ob odnom metode reshenia uravnenia diffuzii (teploprovodnosti), *Inzh.-Fiz. Zh.* N 5 (1959).
2. O. S. BERLYAND and L. A. SOKOLOVSKAYA, O funktsiakh i^n erf $c\,x$, *Inzh.-Fiz. Zh.* N 11 (1958).
3. O. S. BERLYAND, R. I. GAVRILOVA and A. P. PRUDNIKOV, O funktsiakh, udovletvoryaushchikh differentsialnomu uravneniu $y'' \pm 2xy' \pm 2ny = 0$, *Inzh.-Fiz. Zh.* N 9 (1960).
4. O. S. BERLYAND, *Dokl. Akad. Nauk S.S.S.R.* **124**, N 3 (1959).
5. VAN DER POHL and BREMMER, *Operatsionnoe ischislenie na osnove dvukhstoronnego preobrazovaniya Laplasa*. Izd. Inostrannoi Lit., Moscow.

Abstract—A method of solution of the heat conduction (diffusion) equation is developed for a non-stationary three-dimensional problem with convection. A theoretical substantiation of the method is present for the case of a horizontal (longitudinal) velocity component of medium displacement being an arbitrary polynomial relative to the vertical co-ordinate. The solution for the equation is based on the operational method and has the form of a series arranged according to the powers of the generalized Laplace parameter.

Résumé—La méthode de résolution de l'équation de conduction de la chaleur (diffusion) est étudiée dans le cas d'un problème transitoire, tridimensionnel avec convection. Une application théorique de la méthode est présentée dans le cas où la composante de la vitesse horizontale (longitudinale) du déplacement moyen est un polynôme quelconque de la coordonnée verticale. La solution de l'équation est fondée sur la méthode opérationnelle et a la forme d'une série ordonnée suivant les puissances du paramètre généralisé de Laplace.

Zusammenfassung—Die Lösungsmethode der Wärmeleitungsgleichung (Diffusion) wird auf das instationäre dreidimensionale Problem mit Konvektion ausgedehnt. Theoretisch bewiesen wird diese Methode für den Fall, dass die horizontale (longitudinale) Geschwindigkeitskomponente von mittlerer Grösse als willkürliche Polynomfunktion der Vertikal-Koordinate gesetzt werden kann. Die Lösung der Gleichung ist in Form einer Reihe mit steigenden Potenzen des verallgemeinerten Laplace-Parameters angegeben.

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